

# Poncelet Surprises in the Euclidean Plane

Ronaldo A. Garcia  
Dan S. Reznik



33<sup>o</sup> Colóquio  
Brasileiro de  
Matemática

# **Poncelet Surprises in the Euclidean Plane**

## **Poncelet Surprises in the Euclidean Plane**

Primeira impressão, julho de 2021

Copyright © 2021 Ronaldo A. Garcia e Dan S. Reznik.

Publicado no Brasil / Published in Brazil.

**ISBN** 978-65-89124-43-6

**MSC** (2020) Primary: 37M05, Secondary: 14H70, 37C83, 51M15, 51N20, 14Q05

**Coordenação Geral**

Carolina Araujo

**Produção** Books in Bytes

**Capa** Izabella Freitas & Jack Salvador

**Realização da Editora do IMPA**

**IMPA**

Estrada Dona Castorina, 110

Jardim Botânico

22460-320 Rio de Janeiro RJ

[www.impa.br](http://www.impa.br)

[editora@impa.br](mailto:editora@impa.br)

# Preface

---

Since the discovery of Poncelet's porism in the 1810s, a steady stream of proofs has been put forth, drawing upon the ever-evolving language and abstraction of mathematics. These started in the 19th century with Poncelet's own synthetic/analytic proof, passing through Jacobi's treatment with elliptic functions, all the way to our era where the phenomenon is understood on an abstract torus. See Del Centina (2016a,b) for the historical background.

Indeed, for the past 200 years, the focus has been on refining proofs and understanding ramifications of the porism with respect to other areas of Mathematics. One consequence has been that the ambient, dynamic planar geometry of Poncelet polygons has been mostly unexplored.

In this book we take this less-traveled road, i.e., utilizing tools of interactive simulation, we set off to discover curious phenomena manifested by Poncelet polygons in the Euclidean plane. These include invariant metric quantities, the shape of loci of certain points, etc. Luckily, we have stumbled upon many interesting phenomena. Whenever possible, we illustrate the results with pictures and/or animations. To further engage the reader, we propose many exercises and research questions.

This research started in 2011 following lively conversations with Jair Koiller about the path of light rays in an ellipse. This resulted in several Mathematica simulations and a few videos uploaded to YouTube. After an 8-year hiatus, we resumed the work in early 2019 following a few very auspicious events: (i) one of the authors learned other mathematicians had watched our videos and published proofs of phenomena therein, (ii) Sergei Tabachnikov's invited us to publish an



article (jointly with Jair Koiller) in the *Mathematical Intelligencer*, and (ii) our expository talk at IMPA's 32nd colloquium of Brazilian mathematics (see [this video](#)). Following this process our research sped up and we ended up producing dozens of papers and hundreds of experimental videos, which form the basis of this book.

We are indebted to several mathematicians and friends who have answered hundreds of our emails, and shared with us much needed insights. Alphabetically: Arseniy Akopyan, Michael Bialy, Ana Chávez-Caliz, Mário Jorge Carneiro, Manish Chakrabarti, Ethan Cotterill, Marcos Craizer, Iverton Darlan, Carlos Esperança, Robert Ferréol, Corentin Fierobe, Sergey Galkin, Liliana Gheorghe, Bernard Gibert, João Gondim, Darij Grinberg, Mark Helman, Daniel Jaud, Clark Kimberling, Jair Koiller, Dominique Laurain, Nicholas McDonald, Peter Moses, Oliver Nash, Boris Odehnal, Matt Perlmutter, Pedro Roitman, Olga Romaskevich, Richard Schwartz, Hellmuth Stachel, Sergei Tabachnikov, Israel Vainsencher, Daniel Weller, Jorge Zubelli, and others.

We also thank IMPA for the opportunity to publish this book supporting our course in the 33rd Colloquium of Brazilian Mathematics (2021), and Paulo Ney de Souza for his encouragement, and editorial support.

Ronaldo Garcia & Dan Reznik

Goiânia & Rio de Janeiro, Brazil

July, 2021

# Contents

---

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Poncelet preliminaries . . . . .	1
1.2	The elliptic billiard . . . . .	3
1.3	Focusing on 3-periodics . . . . .	3
1.4	Asking simple questions . . . . .	5
1.5	On to more confocal results . . . . .	6
1.6	Branching out to non-confocal families . . . . .	7
1.7	Analysis methods . . . . .	7
1.8	Related work . . . . .	9
1.9	Book organization . . . . .	9
<b>2</b>	<b>Confocal Pair</b>	<b>11</b>
2.1	Preliminaries . . . . .	11
2.2	Caustic semiaxes . . . . .	12
2.3	Incenter and excenter loci . . . . .	13
2.4	A stationary point . . . . .	15
2.5	Conserved quantities . . . . .	17
2.6	An interpretation for Darboux's constant . . . . .	20
2.7	Confocal vertex parametrization . . . . .	20
	2.7.1 Standard . . . . .	20
	2.7.2 Jacobi's universal measure . . . . .	21
2.8	Exercises . . . . .	25
2.9	Research questions . . . . .	26

<b>3</b>	<b>Concentric, Axis-Parallel (CAP)</b>	<b>28</b>
3.1	Excentral family . . . . .	28
3.2	Incircle family . . . . .	32
3.2.1	Confocal affine image . . . . .	33
3.3	Circumcircle family . . . . .	34
3.3.1	Confocal affine image . . . . .	36
3.4	Homothetic family . . . . .	37
3.5	Dual family . . . . .	39
3.6	Vertex parametrization for a generic CAP pair . . . . .	41
3.7	Summary . . . . .	42
3.8	Exercises . . . . .	44
3.9	Research questions . . . . .	46
<b>4</b>	<b>Non-concentric, Axis-Parallel (NCAP)</b>	<b>48</b>
4.1	Poristic family (Bicentric triangles) . . . . .	48
4.2	Poristic excentrals . . . . .	55
4.3	The Brocard porism . . . . .	59
4.3.1	A digression: equilateral isodynamic pedals . . . . .	67
4.4	Vertex parametrization . . . . .	69
4.4.1	Poristic family . . . . .	69
4.4.2	Poristic excentrals . . . . .	70
4.4.3	Brocard porism . . . . .	70
4.5	Summary . . . . .	71
4.6	Exercises . . . . .	72
4.7	Research questions . . . . .	73
<b>5</b>	<b>Confocal Loci</b>	<b>75</b>
5.1	Kimberling centers with elliptic loci . . . . .	76
5.2	When billiard 3-periodics are obtuse . . . . .	78
5.3	Quartic locus of the symmedian point $X_6$ . . . . .	79
5.4	Feuerbach point and its anticomplement . . . . .	82
5.5	A locus with singularities . . . . .	83
5.6	A self-intersecting locus . . . . .	85
5.7	A non-compact locus . . . . .	85
5.8	A golden locus . . . . .	88
5.9	When the billiard is swept non-monotonically . . . . .	88
5.10	The dance of the swans . . . . .	90
5.11	Locus of vertices of derived triangles . . . . .	93

5.12	Locus triple winding . . . . .	98
5.13	Exercises . . . . .	98
5.14	Research questions . . . . .	101
<b>6</b>	<b>Loci in CAP Pairs</b>	<b>103</b>
6.1	Incircle family . . . . .	104
6.2	Circumcircle family . . . . .	105
6.3	Homothetic family . . . . .	107
6.3.1	Four circular loci . . . . .	108
6.3.2	Loci of the Brocard points . . . . .	110
6.3.3	First Brocard triangle: vertex locus . . . . .	111
6.3.4	Loci of Fermat and isodynamic equilaterals . . . . .	112
6.4	Dual family . . . . .	113
6.5	Excentral family . . . . .	113
6.6	Summary . . . . .	113
6.6.1	Loci types, CAP families . . . . .	117
6.6.2	Loci types, NCAP families . . . . .	118
6.7	Exercises . . . . .	119
6.8	Research questions . . . . .	121
<b>7</b>	<b>Analyzing Loci</b>	<b>122</b>
7.1	When are loci algebraic? . . . . .	123
7.2	Review: Blaschke products . . . . .	125
7.3	Locus of the incenter in a generic pair . . . . .	128
7.4	Loci in generic nested ellipses . . . . .	132
7.5	Circular loci in the circumcircle family . . . . .	137
7.6	Elliptic loci in the confocal pair . . . . .	141
7.7	Exercises . . . . .	146
7.8	Research questions . . . . .	147
<b>8</b>	<b>The Focus-Inversive Family</b>	<b>148</b>
8.1	Non-Ponceletian . . . . .	148
8.2	A stationary point . . . . .	149
8.3	Billiard-like invariants . . . . .	151
8.4	The rotating billiard table . . . . .	151
8.5	Invariant area product . . . . .	152
8.6	Circular loci galore! . . . . .	155

8.7	A rule for circular loci? . . . . .	156
8.7.1	Centroidal loci: a tale of three circles . . . . .	157
8.8	A focus-inversive Doppelgänger . . . . .	159
8.9	Exercises . . . . .	162
8.10	Research questions . . . . .	163
<b>9</b>	<b>A Locus Visualization App</b>	<b>164</b>
9.1	Main ellipse and animation controls . . . . .	165
9.1.1	Convenience animation controls . . . . .	166
9.2	Channel controls . . . . .	167
9.3	Choosing a triangle family . . . . .	167
9.3.1	Poncelet families . . . . .	167
9.3.2	Ellipse “mounted” . . . . .	169
9.4	Triangle type . . . . .	171
9.4.1	Standard triangles . . . . .	171
9.4.2	Exotic triangles . . . . .	171
9.4.3	Inversive triangles . . . . .	172
9.5	Locus type . . . . .	173
9.5.1	Centers and vertices . . . . .	173
9.5.2	Envelopes . . . . .	174
9.5.3	Bicentric pairs . . . . .	174
9.6	Triangle center . . . . .	174
9.7	Cevians, pedals, & Co. . . . .	175
9.7.1	Traditional . . . . .	175
9.7.2	Inversive . . . . .	175
9.7.3	Reflexive . . . . .	177
9.7.4	Triangulated . . . . .	177
9.8	Notable circles . . . . .	177
9.8.1	Ellipse-affixed circles . . . . .	178
9.8.2	Central circles . . . . .	178
9.9	Inversive transformations with respect to a circle . . . . .	180
9.10	Conic and invariant detection . . . . .	181
9.10.1	Curve type . . . . .	181
9.10.2	Detection of metric invariants . . . . .	181
9.11	The tandem bar . . . . .	183
9.12	Odds & ends . . . . .	185
9.12.1	Ellipse, locus tange, and animation background . . . . .	185
9.12.2	Resetting the UI and centering the animation . . . . .	185

9.12.3	Setting the locus color . . . . .	188
9.12.4	Collapsing the locus control area . . . . .	188
9.13	Artsy loci . . . . .	188
9.14	Sharing and exporting . . . . .	189
9.15	Jukebox playback . . . . .	192

**A Notes in Triangle Geometry 194**

A.1	Trilinear coordinates . . . . .	194
A.2	More calculations with distances . . . . .	195
A.3	Barycentric coordinates . . . . .	197
A.4	Conversion to and from cartesians . . . . .	197
A.5	Triangle centers . . . . .	198
A.6	Selected triangle centers . . . . .	198
A.7	Some derived triangles . . . . .	201
A.8	The (first) Brocard triangle . . . . .	205
A.9	Pedal and antipedal triangles . . . . .	206
A.10	Cevian triangle . . . . .	209
A.11	Perspective triangles . . . . .	209
A.12	Polar triangle . . . . .	209
A.13	Circumconic . . . . .	210
A.14	Inconic . . . . .	211
A.15	Brocard inellipse . . . . .	211
A.16	Ceva conjugate . . . . .	212
A.17	Isogonal conjugation . . . . .	212
A.18	Isotomic conjugation . . . . .	214
A.19	The Euler line . . . . .	216
A.20	Circumconic and inconic . . . . .	217
A.21	Billiard notes . . . . .	218
A.22	Exercises . . . . .	219

**B Jacobi Elliptic Functions 225**

B.1	Jacobi elliptic integral and inverse . . . . .	225
B.2	Jacobi elliptic functions . . . . .	225
B.3	Basic identities . . . . .	226
B.4	Connection with differential equations . . . . .	227
B.5	Inverse Jacobi elliptic functions . . . . .	227
B.6	Complex plane extension . . . . .	228



<b>C Ellipse-Mounted Brocard loci</b>	<b>229</b>
C.1 Circular sweep, one vertex at center . . . . .	229
C.2 Circular sweep, two vertices at 90-degrees . . . . .	230
C.3 Circular sweep, antipodal vertices . . . . .	230
C.4 Ellipse sweep, two vertices at major endpoints . . . . .	231
C.5 Elliptic sweep, vertices on major axis . . . . .	231
<b>Bibliography</b>	<b>236</b>
<b>Index</b>	<b>241</b>
<b>List of Symbols</b>	<b>244</b>

# I

## Introduction

---

### 1.1 Poncelet preliminaries

Poncelet's closure theorem is illustrated in Figure 1.1. It is based on a simple geometric iteration. Given two nested ellipses<sup>1</sup>  $\mathcal{E}$  and  $\mathcal{E}_c$ , pick a point  $P_1$  on the boundary of  $\mathcal{E}$ . Let  $P_2$  be where a ray shot from  $P_1$  along one of the tangents to  $\mathcal{E}_c$  meets  $\mathcal{E}$  again. Repeat this from  $P_2$ , yielding  $P_3$ , etc. This produces a piecewise-linear Poncelet *trajectory*.

For most choices of  $(\mathcal{E}, \mathcal{E}_c)$ , the trajectory will never close, i.e., it will never meet  $P_1$  again. In fact, it will fill a region between the two conics. However, for certain choices<sup>2</sup>, the trajectory will indeed close. Let  $N$ , and integer greater than 2, be the number of steps required for  $P_1$  to be met again. We call such polygonal trajectories “ $N$ -periodic”.

Still referring to Figure 1.1, Poncelet's theorem states that if a trajectory departing from some point  $P_1$  on  $\mathcal{E}$  closes after  $N$  steps, then a *porism* is triggered which prescribes a 1d family of  $N$ -gons: a trajectory departing from *any* other point on the boundary of  $\mathcal{E}$  will also close in  $N$  steps. We say such a pair “admits” a 1d family of  $N$ -periodic trajectories.

---

<sup>1</sup>The theorem is projective, i.e., it works for any pair of conics, nested or not.

<sup>2</sup>Those which satisfy Cayley's conditions, see Dragović and Radnović (2011).

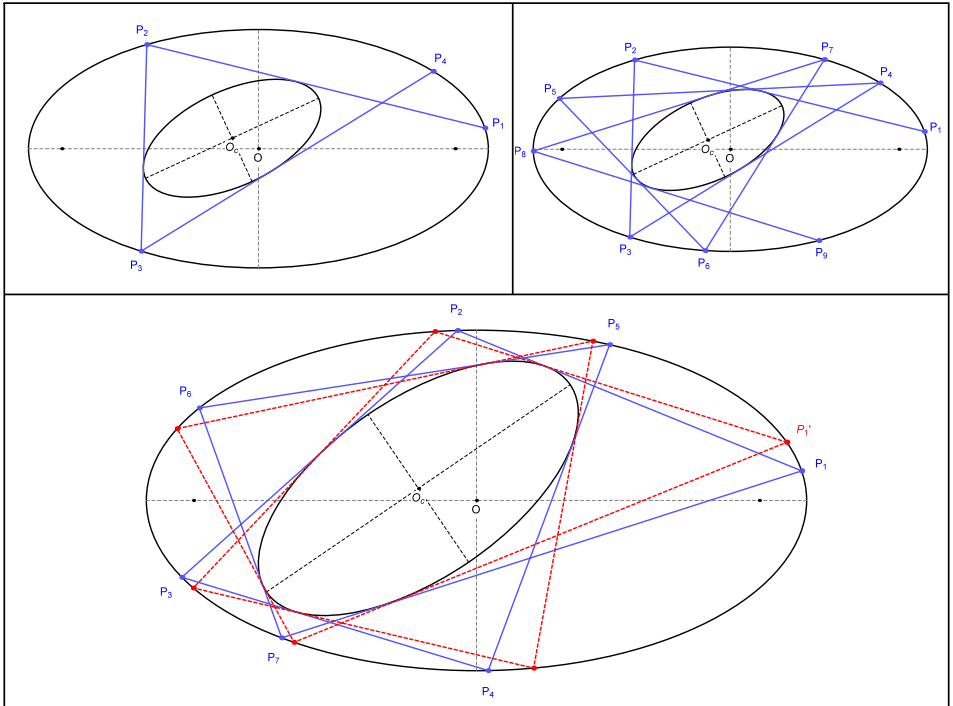


Figure 1.1: **Top left:** 3 Poncelet iterations within a pair of ellipses in general position; their centers are labeled  $O$  and  $O_c$ , respectively. **Top right:** 5 more iterations executed (starting at  $P_4$ ), showing the trajectory is not likely to close. **Bottom:** a new ellipse pair for which an iteration departing from  $P_1$  closes after 7 steps (blue polygon). Poncelet's porism guarantees that if the iteration were to start anywhere else on the outer ellipse, e.g.,  $P_1'$ , it will also yield a closed, 7-gon (dashed red).  
 Video, Live

## Títulos Publicados — 33º Colóquio Brasileiro de Matemática

- Geometria Lipschitz das singularidades** – *Lev Birbrair e Edvalter Sena*
- Combinatória** – *Fábio Botler, Maurício Collares, Taísa Martins, Walner Mendonça, Rob Morris e Guilherme Mota*
- Códigos Geométricos** – *Gilberto Brito de Almeida Filho e Saeed Tafazolian*
- Topologia e geometria de 3-variedades** – *André Salles de Carvalho e Rafał Marian Siejakowski*
- Ciência de Dados: Algoritmos e Aplicações** – *Luerbio Faria, Fabiano de Souza Oliveira, Paulo Eustáquio Duarte Pinto e Jayme Luiz Szwarcfiter*
- Discovering Euclidean Phenomena in Poncet Families** – *Ronaldo A. Garcia e Dan S. Reznik*
- Introdução à geometria e topologia dos sistemas dinâmicos em superfícies e além** – *Victor León e Bruno Scárdua*
- Equações diferenciais e modelos epidemiológicos** – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*
- Differential Equation Models in Epidemiology** – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*
- A friendly invitation to Fourier analysis on polytopes** – *Sinai Robins*
- PI-álgebras: uma introdução à PI-teoria** – *Rafael Bezerra dos Santos e Ana Cristina Vieira*
- First steps into Model Order Reduction** – *Alessandro Alla*
- The Einstein Constraint Equations** – *Rodrigo Avalos e Jorge H. Lira*
- Dynamics of Circle Mappings** – *Edson de Faria e Pablo Guarino*
- Statistical model selection for stochastic systems** – *Antonio Galves, Florencia Leonardi e Guilherme Ost*
- Transfer Operators in Hyperbolic Dynamics** – *Mark F. Demers, Niloofar Kiamari e Carlangelo Liverani*
- A Course in Hodge Theory Periods of Algebraic Cycles** – *Hossein Movasati e Roberto Villaflor Loyola*
- A dynamical system approach for Lane–Emden type problems** – *Liliane Maia, Gabrielle Nornberg e Filomena Pacella*
- Visualizing Thurston’s Geometries** – *Tiago Novello, Vinícius da Silva e Luiz Velho*
- Scaling Problems, Algorithms and Applications to Computer Science and Statistics** – *Rafael Oliveira e Akshay Ramachandran*
- An Introduction to Characteristic Classes** – *Jean-Paul Brasselet*



Instituto de  
Matemática  
Pura e Aplicada

ISBN 978-65-89124-43-6



9 786589 124436

