Poncelet Surprises in the Euclidean Plane

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Preface

Since the discovery of Poncelet's porism in the 1810s, a steady stream of proofs has been put forth, drawing upon the ever-evolving language and abstraction of mathematics. These started in the 19th century with Poncelet's own synthetic/analytic proof, passing through Jacobi's treatment with elliptic functions, all the way to our era where the phenomenon is understood on an abstract torus. See Del Centina (2016a,b) for the historical background.

Indeed, for the past 200 years, the focus has been on refining proofs and understanding ramifications of the porism with respect to other areas of Mathematics. One consequence has been that the ambient, dynamic planar geometry of Poncelet polygons has been mostly unexplored.

In this book we take this less-traveled road, i.e., utilizing tools of interactive simulation, we set off to discover curious phenomena manifested by Poncelet polygons in the Euclidean plane. These include invariant metric quantities, the shape of loci of certain points, etc. Luckily, we have stumbled upon many interesting phenomena. Whenever possible, we illustrate the results with pictures and/or animations. To further engage the reader, we propose many exercises and research questions.

This research started in 2011 following lively conversations with Jair Koiller about the path of light rays in an ellipse. This resulted in several Mathematica simulations and a few videos uploaded to YouTube. After an 8-year hiatus, we resumed the work in early 2019 following a few very auspicious events: (i) one of the authors learned other mathematicians had watched our videos and published proofs of phenomena therein, (ii) Sergei Tabachnikov's invited us to publish an article (jointly with Jair Koiller) in the Mathematical Intelligencer, and (ii) our expository talk at IMPA's 32nd colloquium of Brazilian mathematics (see this video). Following this process our research sped up and we ended up producing dozens of papers and hundreds of experimental videos, which form the basis of this book.

We are indebted to several mathematicians and friends who have answered hundreds of our emails, and shared with us much needed insights. Alphabetically: Arseniy Akopyan, Michael Bialy, Ana Chávez-Caliz, Mário Jorge Carneiro, Manish Chakrabarti, Ethan Cotterill, Marcos Craizer, Iverton Darlan, Carlos Esperança, Robert Ferréol, Corentin Fierobe, Sergey Galkin, Liliana Gheorghe, Bernard Gibert, João Gondim, Darij Grinberg, Mark Helman, Daniel Jaud, Clark Kimberling, Jair Koiller, Dominique Laurain, Nicholas McDonald, Peter Moses, Oliver Nash, Boris Odehnal, Matt Perlmutter, Pedro Roitman, Olga Romaskevich, Richard Schwartz, Hellmuth Stachel, Sergei Tabachnikov, Israel Vainsencher, Daniel Weller, Jorge Zubelli, and others.

We also thank IMPA for the opportunity to publish this book supporting our course in the 33rd Colloquium of Brazilian Mathematics (2021), and Paulo Ney de Souza for his encouragement, and editorial support.

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1 Introduction

1.1 Poncelet preliminaries

Poncelet's closure theorem is illustrated in Figure 1.1. It is based on a simple geometric iteration. Given two nested ellipses[1] $\mathcal{E}$ and $\mathcal{E}_c$, pick a point P_1 on the boundary of $\mathcal{E}$. Let P_2 be where a ray shot from P_1 along one of the tangents to $\mathcal{E}_c$ meets $\mathcal{E}$ again. Repeat this from P_2, yielding P_3, etc. This produces a piecewise-linear Poncelet *trajectory*.

For most choices of $(\mathcal{E}, \mathcal{E}_c)$, the trajectory will never close, i.e., it will never meet P_1 again. In fact, it will fill a region between the two conics. However, for certain choices [2], the trajectory will indeed close. Let N, and integer greater than 2, be the number of steps required for P_1 to be met again. We call such polygonal trajectories "N-periodic".

Still referring to Figure 1.1, Poncelet's theorem states that if a trajectory departing from some point P_1 on $\mathcal{E}$ closes after N steps, then a *porism* is triggered which prescribes a 1d family of N-gons: a trajectory departing from *any* other point on the boundary of $\mathcal{E}$ will also close in N steps. We say such a pair "admits" a 1d family of N-periodic trajectories.

[1] The theorem is projective, i.e., it works for any pair of conics, nested or not.

[2] Those which satisfy Cayley's conditions, see Dragović and Radnović (2011).

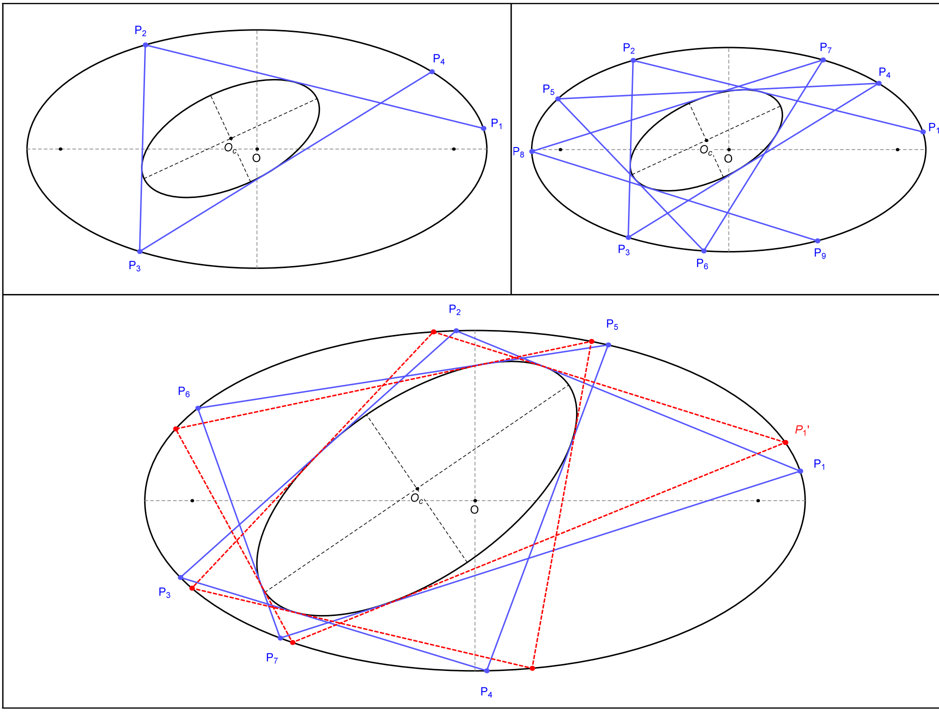

Figure 1.1: **Top left**: 3 Poncelet iterations within a pair of ellipses in general position; their centers are labeled O and O_c, respectively. Top right: 5 more iterations executed (starting at P_4), showing the trajectory is not likely to close. **Bottom:** a new ellipse pair for which an iteration departing from P_1 closes after 7 steps (blue polygon). Poncelet's porism guarantees that if the iteration were to start anywhere else on the outer ellipse, e.g., P_1', it will also yield a closed, 7-gon (dashed red). Video, Live

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