A friendly invitation to Fourier analysis on polytopes

Sinai Robins

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33º Colóquio Brasileiro de Matemática

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Primeira impressão, julho de 2021

Publicado no Brasil / Published in Brazil.

ISBN 978-65-89124-35-1
MSC (2020) Primary: 52B20, Secondary: 52C07, 11H06, 52C22, 44A05, 05A15

Coordenação Geral Carolina Araujo

Produção Books in Bytes **Capa** Izabella Freitas & Jack Salvador

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ
www.impa.br
editora@impa.br

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Acknowledgements

The famous saying "no man is an island" is doubly-true in Mathematics, and indeed I've had the good fortune to know and learn from many interesting people, concerning the contents of this book. Special thanks goes to Ricardo Diaz, my first collaborator along these topics. I would like to thank the following people, from the bottom of my heart, for their valuable input and interesting discussions about some of these topics over the years:

Artur André, Christine Bachoc, Tamar Bar, Imre Bárány, Alexander Barvinok, Matthias Beck, Luca Brandolini, Michel Brion, Henry Cohn, Leonardo Colzani, Pierre Deligne, Jesús A. De Loera, Michel Faleiros, Lenny Fukshansky, Nick Gravin, Martin Henk, Didier Henrion, Roberto Hirata Junior, Jeffrey Hoffstein, Alex Iosevich, Marvin Knopp, Mihalis Kolountzakis, Matthias Köppe, Jean Bernard Lasserre, Nhat Le Quang, Rafael Zuolo Coppini Lima, Fabrício Caluza Machado, Romanos Malikiosis, Máté Matolci, Tyrrell McAllister, Victor Moll, Mel Nathanson, James Pommersheim, Jim Propp, Thales Paiva, Jill Pipher, Jorge Luis Ramírez Alfonsín, Bruce Reznick, Tiago Royer, Nicolas Salter, Gervásio Santos, Richard Schwartz, Dima Shiryaev, Joseph Silverman, Richard Stanley, Christophe Vignat, Sergei Tabachnikov, Giancarlo Travaglini, Kevin Woods, Günter Ziegler, Chuanming Zong.

Preface

Figure 1: Joseph Fourier

What is a Fourier transform? Why is it so useful? How can we apply Fourier transforms and Fourier series - which were originally used by Fourier to study heat diffusion - in order to better understand topics in discrete and combinatorial geometry, number theory, and sampling theory?

To begin, there are some useful analogies: imagine that you are drinking a milkshake (lactose free), and you want to know the ingredients of your tasty drink. You would need to filter out the shake into some of its most basic components. This decomposition into its basic ingredients may be thought of as a sort of "Fourier transform of the milkshake". Once we understand each of the ingredients, we will also be able to restructure these ingredients in new ways, to form many other types of tasty goodies. To move the analogy back into mathematical language, the milkshake represents a function, and each of its basic ingredients represents for us the basis of sines and cosines; we may also think of a basic ingredient more compactly as a complex exponential $e^{2\pi i n x}$, for some $n \in \mathbb{Z}$. Composing these basic ingredients together in a new way represents a Fourier series.

Mathematically, one of the most basic kinds of milkshakes is the indicator function of the unit interval, and to break it down into its basic components, mathematicians, Engineers, Computer scientists, and Physicists have used the sinc function (since the 1800's):

$$\mathrm{sinc}(z) := \frac{\sin(\pi z)}{\pi z}$$

with great success, because it happens to be the Fourier transform of the unit interval $[-\frac{1}{2}, \frac{1}{2}]$:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i z x} dx = \mathrm{sinc}(z),$$

as we will compute shortly in identity (2.5). Somewhat surprisingly, comparatively little energy has been given to some of its higher dimensional extensions, namely those extensions that arise naturally as Fourier transforms of polytopes.

One motivation for this book is to better understand how this 1-dimensional function – which has proved to be extremely powerful in applications – extends to higher dimensions. Namely, we will build various mathematical structures that are motivated by the question:

What is the Fourier transform of a polytope?

Of course, we will ask "how can we apply it"? An alternate title for this book might have been:

We're taking Poisson summation and Fourier transforms of polytopes for a very long ride....

Historically, sinc functions were used by Shannon (as well as Hardy, Kotelnikov, Nyquist, and Whittaker) when he published his seminal work on sampling theory and information theory.

In the first part of this book, we will learn how to use the technology of Fourier transforms of polytopes in order to build the (Ehrhart) theory of integer point enumeration in polytopes, to prove some of Minkowski's theorems in the geometry of numbers, and to understand when a polytope tiles Euclidean space by translations.

In the second portion of this book, we give some applications to active research areas which are sometimes considered more applied, including the sphere packing problem, and the angle polynomial of a polytope.

There are also current research developments of the material developed here, to the learning of deep neural networks. In many applied scientific areas, in particular radio astronomy, computational tomography, and magnetic resonance imaging, a frequent theme is the reconstruction of a function from knowledge of its Fourier transform. Somewhat surprisingly, in various applications we only require very partial/sparse knowledge of its Fourier transform in order to reconstruct the required function, which may represent an image or a signal.

There is a rapidly increasing amount of research focused in these directions in recent years, and it is therefore time to put many of these new findings in one place, making them accessible to a general scientific reader. The fact that the sinc function is indeed the Fourier transform of the 1-dimensional line segment $[-\frac{1}{2}, \frac{1}{2}]$, which is a 1-dimensional polytope, gives us a first hint that there is a deeper link between the geometry of a polytope and the analysis of its Fourier transform.

Indeed one reason that sampling and information theory, as initiated by Claude Shannon, works so well is precisely because the Fourier transform of the unit interval has this nice form, and even more so because of the existence of the Poisson summation formula.

The approach we take here is to gain insight into how the Fourier transform of a polytope can be used to solve various specific problems in discrete geometry, combinatorics, optimization, and approximation theory:

(a) Analyze tilings of Euclidean space by translations of a polytope

(b) Give wonderful formulas for volumes of polytopes

(c) Compute discrete volumes of polytopes, which are combinatorial approximations to the continuous volume

(d) Introduce the geometry of numbers, via Poisson summation

(e) Optimize sphere packings, and get bounds on their optimal densities

Let's see at least one direction that quickly motivates the study of Fourier transforms. In particular, we often begin with simple-sounding problems that arise naturally in combinatorial enumeration, discrete and computational geometry, and number theory.

Throughout, an **integer point** is any vector $v := (v_1, \dots, v_d) \in \mathbb{R}^d$, all of whose coordinates v_j are integers. In other words, v belongs to the integer lattice $\mathbb{Z}^d$. A **rational point** is a point m whose coordinates are rational numbers, in other words $m \in \mathbb{Q}^d$. We define the **Fourier transform** of a function $f(x)$:

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx, \tag{1}$$

defined for all $\xi \in \mathbb{R}^d$ for which the latter integral converges, and where we use the standard inner product $\langle a, b \rangle := a_1 b_1 + \cdots + a_d b_d$. We will also use the notation $\mathcal{F}(f)$ for the Fourier transform of f, which is useful in some typographical contexts, for example when considering $\mathcal{F}^{-1}(f)$.

Now we can introduce one of the main objects of study in this book, the **Fourier transform of a polytope** $\mathcal{P}$, defined by:

$$\hat{1}_{\mathcal{P}}(\xi) := \int_{\mathbb{R}^d} 1_{\mathcal{P}}(x) e^{-2\pi i \langle \xi, x \rangle} dx = \int_{\mathcal{P}} e^{-2\pi i \langle \xi, x \rangle} dx, \tag{2}$$

where the function $1_{\mathcal{P}}(x)$ is the **indicator function** of $\mathcal{P}$, defined by

$$1_{\mathcal{P}}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{P} \\ 0 & \text{if not.} \end{cases}$$

Thus, the words "Fourier transform of a polytope $\mathcal{P}$" will always mean the Fourier transform of the indicator function of $\mathcal{P}$.

The **Poisson summation formula**, named after Siméon Denis Poisson, tells us that for any "sufficiently nice" function $f : \mathbb{R}^d \to \mathbb{C}$ we have:

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi).$$

In particular, if we were to naively set $f(n) := 1_{\mathcal{P}}(n)$, the indicator function of a polytope $\mathcal{P}$, then we would get:

$$\sum_{n \in \mathbb{Z}^d} 1_{\mathcal{P}}(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\mathcal{P}}(\xi), \tag{3}$$

which is technically false in general due to the fact that the indicator function $1_{\mathcal{P}}$ is a discontinuous function on $\mathbb{R}^d$.

However, this technically false statement is very useful! We make this claim because it helps us build intuition for the more rigorous statements that are true, and which we study in later chapters. For applications to discrete geometry, we are interested in the number of integer points in a closed convex polytope $\mathcal{P}$, namely $|\mathcal{P} \cap \mathbb{Z}^d|$. The combinatorial-geometric quantity $|\mathcal{P} \cap \mathbb{Z}^d|$ may be regarded as a **discrete volume** for $\mathcal{P}$. From the definition of the indicator function of a polytope, the left-hand-side of (3) counts the number of integer points in $\mathcal{P}$, namely we have by definition

$$\sum_{n \in \mathbb{Z}^d} 1_{\mathcal{P}}(n) = |\mathcal{P} \cap \mathbb{Z}^d|. \tag{4}$$

On the other hand, the right-hand-side of (3) allows us to compute this discrete volume of $\mathcal{P}$ in a new way. This is great, because it opens a wonderful window of computation for us in the following sense:

$$|\mathcal{P} \cap \mathbb{Z}^d| = \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\mathcal{P}}(\xi). \tag{5}$$

We notice that for the $\xi = 0$ term, we have

$$\hat{1}_{\mathcal{P}}(0) := \int_{\mathbb{R}^d} 1_{\mathcal{P}}(x) e^{-2\pi i \langle 0, x \rangle} dx = \int_{\mathcal{P}} dx = \text{vol}(\mathcal{P}), \tag{6}$$

and therefore the discrepancy between the continuous volume of $\mathcal{P}$ and the discrete volume of $\mathcal{P}$ is

$$|\mathcal{P} \cap \mathbb{Z}^d| - \text{vol}(\mathcal{P}) = \sum_{\xi \in \mathbb{Z}^d - \{0\}} \hat{1}_{\mathcal{P}}(\xi), \tag{7}$$

showing us very quickly that indeed $|\mathcal{P} \cap \mathbb{Z}^d|$ is a discrete approximation to the classical Lebesgue volume $\text{vol}(\mathcal{P})$, and pointing us to the task of finding ways to evaluate the transform $\hat{1}_P(\xi)$. From the trivial but often very useful identity

$$\hat{1}_{\mathcal{P}}(0) = \text{vol}(\mathcal{P}),$$

we see another important motivation for this book: the Fourier transform of a polytope is a very **natural extension of volume**. Computing the volume of a polytope $\mathcal{P}$ captures a bit of information about $\mathcal{P}$, but we also lose a lot of information.

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ISBN 978-65-89124-35-1

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