The Einstein Constraint Equations

Rodrigo Avalos
Jorge H. Lira

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Preface

This book was written as lecture notes for a mini-course on the Einstein constraint equations (ECE) delivered in the 33^{rd} Brazilian Colloquium of Mathematics. It is directed to a wide audience of students and researchers interested in the overlap of Riemannian geometry, geometric analysis and physics. The focus of these notes is to provide a quite thorough description of the so-called conformal method, which translates the geometric ECE into an elliptic system of partial differential equations (PDEs) in a nearly self contained presentation, ranging from classical results to recent progress. This is a subject which intersects several traditional problems in geometric analysis, such as scalar curvature prescription and the Yamabe problem, and which has its roots in the evolution problem of initial data in general relativity (GR). As such, it has become a whole area of research within mathematical GR and its intersection with classic problems in geometric analysis has produced plenty of feedback between these areas. We shall assume the reader is familiarised with classical topics and language in both differential geometry and Riemannian geometry as well as with standard functional analysis, which is used within PDE theory. We do not assume the reader to be necessarily acquainted with elliptic equations and, with this in mind, we have built an appendix compiling the necessary tools which are used in the core of the book. Also, some of the most recurrent functional analytic tools are also compiled within the first appendix of the book, with emphasis on Sobolev space theory, which provides the reader with all the necessary tools to follow the main chapters without many outside references.

The organisation of the book is intended to deliver a clear exposition highlighting the relevance of the analysis of the ECE, their many subtleties and an up-to-date presentation of the results available in this area. In doing so, we have been inspired by recent literature in the subject, most notably the monograph of Choquet-Bruhat (2009) and several recent papers such as Holst, Nagy, and Tsogtgerel (2009) and Maxwell (2005a,b, 2009). We have gone through the classical constant mean curvature (CMC) classifications on closed manifolds originated in Isenberg (1995), but putting them in light of these recent advances, and thus presented them in low regularity and also contemplating non-vacuum situations. Along these lines, we have complemented several of these recent references. Furthermore, we have made emphasis in the analysis on asymptotically Euclidean (AE) manifolds, incorporating boundary value problems, and, as a novelty in a book on the subject, we have introduced recent advances on far-from-CMC existence of solutions.

Chapter 1 is meant to be an introduction to general relativity with the objective of setting up the problem, reviewing the context in which the ECE arise, producing some intuitions and motivating the analysis of boundary problems associated to black hole solutions as well as highly coupled systems exemplified by charged fluids. Also, in this chapter we set most of our notational conventions. The topics here included are standard for any specialist in GR, but are intended to serve as a good introduction for the unfamiliarised reader, from whom we do not assume any sophisticated knowledge of physics.

Chapter 2 starts by presenting the conformal method and translating the ECE into a geometric elliptic system. In doing so, we contemplate very general situations which incorporate the conformal formulation of the Gauss–Codazzi constraints coupled with a further electromagnetic constraint. Then, we start our analysis with the CMC case admitting sources which allow the system to be fully decoupled and thus the core of the analysis is devoted to the associated Lichnerowicz equation. During this chapter we will give a near state-of-the-art presentation of this problem following Maxwell (2005a), and therefore establishing an L^p-low-regularity complete CMC classification on closed manifolds which incorporates several physical sources. In the process of doing so, we shall review results concerning the Yamabe classification in this low regularity setting.

In Chapter 3, we move to the analysis of the Lichnerowicz equation on AE manifolds and introduce boundary value problems which model black hole initial data within the conformal method. We deliver a quite self-contained presentation of the necessary elliptic theory on AE manifolds, which appeals to analysis on weighted Sobolev spaces. We introduce the basic machinery associated to these problems merely assuming basic acquaintance of the reader with the corresponding theory on compact manifolds. We shall present a wide variety of results associated to classical papers such as Bartnik (1986), Cantor (1981), Choquet-Bruhat and Christodoulou (1981), Lockhart (1981), McOwen (1979), and Nirenberg and Walker (1973). After doing this, the main results related to the ECE will be an exposition of Maxwell (2005b).

Chapter 4 is devoted to a presentation of far-from-CMC results. These are quite recent advances in the analysis of the ECE which rely on the application of some fixed-point-theorem ideas and make use of the full machinery developed in previous chapters. We shall first review some near CMC results, attainable through implicit function techniques, and then provide a presentation of the far-from-CMC results established in Maxwell (2009), which followed the pioneering work of Holst, Nagy, and Tsogtgerel (2009). These results concern the coupled ECE in vacuum on closed manifolds. Finally, we will move towards the analysis of the ECE for a charged perfect fluid on AE manifolds with black hole boundary data and present the far-from-CMC results of Avalos and Lira (2019).

Although during the main core of the text the reader is assumed to be familiarised with elliptic theory on closed manifold, in order to provide a self-contained presentation, we have provided most of the necessary tools within two appendixes, where the reader can consult all the results which are used in the main chapters. The first of these appendixes is concerned with some functional analytic tools while the second one with elliptic theory. Since these are extensive areas on their own right, our presentation has been more expository in nature, attempting to provide the reader with full proofs whenever possible, and, when the details exceed the scope of these notes, provide full references as well as the basic intuitions on the ideas behind the actual proofs.

We expect these notes to help researchers within theoretical physics and pure and applied mathematics to become familiarised with some of the many interesting problems in the analysis of the ECE. Some related topics had to be left outside due to time constraints for our course, but a thorough list of references has been provided which the interested reader can use to substantially expand the scope of this book.

Contents

1 Introduction to general relativity

The objective of these notes is to analyse the so-called Einstein constraint equations (ECE). Naturally, these equation arise in the context of general relativity (GR), more specifically within the initial value formulation of this theory. In particular, solution to the ECE provide us with suitable initial data which we can then evolve into solutions of the space-time Einstein equations. Being GR the best known description of gravitational phenomena up to this date, this alone provides enough motivation for the analysis of the ECE. Nevertheless, from a purely mathematical standpoint, they relate with classical problems in Riemannian geometry, such as scalar curvature prescription problems and related geometric partial differential equation (PDE) problems, which further motivates their analysis.

The aim of this first introductory chapter is to provide a review of the setting where the ECE appear naturally, which is the initial value formulation of GR. In this way we can most effectively motivate their relevance, present model situations of interest and provide intuitions about what is expected to occur in their analysis. Since this is a topic which gathers researchers and students ranging from theoretical physics to geometric analysis, we intend to review several notions which are well-known to experts in each of these areas and should be within reach without too much effort for those who are not. In doing so, we will assume acquaintance with differential geometry as well as Riemannian and semi-Riemannian geometry.[1] As a remark regarding notational conventions, let us highlight that, besides standard notations within geometry, we will use whenever it may be more convenient Einstein's index and summation conventions for coordinate expressions, without further comments.

With the above in mind, the organisation of this chapter will be as follows. First, we will review some definitions and results related specifically to Lorentzian geometry. Our main motivations here will be to introduce enough language from causality theory so that, later on, we can introduce notions such as black hole solutions as well as those of Cauchy hypersurfaces and global hyperbolicity. Then, we will present the skeleton of the theory of special relativity. There, the aim is to introduce notions that will be of relevance in subsequent analysis, such us the basic fields which we shall couple to gravity and for which we shall analyse the existence of appropriate initial data. After this, we will promote this discussion to the context of GR, introducing the Einstein equations and presenting these relevant systems in this general context. Also, we will try to develop some intuitions by presenting a few classical well-known exact solutions. In particular, we intend to provide some rudimentary intuitions concerning black hole solutions by describing the Schwarzschild solution. The objective at this point will be to provide us with the right notions to motivate our discussion on black hole initial data. But, before doing this, we will describe the initial value formulation of general relativity. This, in particular, is a topic which deserves a complete book on its own due to its many subtleties (which the interested reader can actually find, for instance, in Ringström (2009)),[2] and therefore we will merely review those results which are of most relevance to us.

1.1 Some elements of Lorentzian geometry

Let us now introduce some notions related to Lorentzian geometry, most of which can be found in standard references, such as Choquet-Bruhat (2009), Hawking and Ellis (1973), and O'Neill (1983) as well as references therein. Let us first state that,

[1] If needed, the interested reader can consult differential geometric topics in classic textbooks such as J. M. Lee (2013) and Spivak (1999e), Riemannian geometry topics in do Carmo (1992) and Spivak (1999a,b,c,d) and textbooks adapted to semi-Riemannian geometry such as Bishop and Goldberg (1980) and O'Neill (1983).

[2] We further recommend references such as Choquet-Bruhat (2009) for a self-contained presentation of the general problem, as well as Christodoulou and Klainerman (1993) and Klainerman and Nicolò (2003) for issues related to the stability of Minkowski and Dafermos and Rodnianski (2013) for topics related with black hole evolution and stability.

during all this text, manifolds will be assumed to be Hausdorff and second countable and, whenever specifying the dimensionality of a manifold M is relevant, we write M^d for a d-dimensional manifold.

Definition 1.1.1. *A semi-Riemannian manifold* (V, g) *will be called Lorentzian if the metric* g *has constant index equal to* 1.

Let us recall that the index of a symmetric bilinear form on a vector space is defined to be the dimension of the largest subspace where its restriction is negative definite. Therefore, using a local orthonormal frame $\{\theta\}_{\alpha=0}^{n}$, we can write g as

$$g = -\theta^0 \otimes \theta^0 + \sum_{i=1}^{n} \theta^i \otimes \theta^i.$$

As above, we will typically reserve the 0-th direction to be the one over which g is negative definite. In particular, the above shows that one can split tangent vectors $v \in T_pV$ into three cases, which determine their *causal character*.

Definition 1.1.2. *Let* (V, g) *be a Lorentzian manifold and let* $p \in V$. *We will say that a vector* $v \in T_pV$, $v \neq 0$, *is time-like if* $g_p(v, v) < 0$*; light-like (or null) if* $g_p(v, v) = 0$ *and space-like if* $g_p(v, v) > 0$. *Along these lines, we define the light-cone (or null-cone) at* p *as the subset of* T_pV *formed by all the null-vectors.*

Whenever we consider a smooth curve $\gamma : I \subset \mathbb{R} \mapsto V$, if its causal character is constant, that is, if γ' is everywhere time-like, null or space-like, then we will say that γ is time-like, null or space-like respectively. Clearly, an arbitrary curve will not fall into any of these categories since its causal character may change, but, in particular, geodesics have a fixed causal character.[3] In order to clarify some of this terminology, let us anticipate that, in the context of relativity theory, massive particles trace time-like paths in space-time while massless particles (such as photons) trace light-like paths. On the other hand, since no signal can travel faster than light, space-like paths do not represent the dynamics of any kind of particles. In particular, points which are space-like related do not have the possibility of influencing each other. We will therefore say that a curve is *causal* if it is either time-like or light-like.

Let us now highlight the special role played the the following Lorentzian manifold.

[3] During these notes, we will always work with Riemannian (metric compatible and torsion-free) connections, and therefore parallel transport is an isometry.

Definition 1.1.3. *The manifold $\mathbb{R}^{n+1}$ equipped with the Lorentzian metric η given by*

$$\eta = -dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i,$$

where $\{x^\alpha\}_{\alpha=0}^n$ stand for (global) canonical coordinates for $\mathbb{R}^n$, is referred to as the Minkowski space-time, and we will denote it by $\mathbb{M}^{n+1}$.

Therefore, just as Euclidean space is the local model of a Riemannian manifold, in a Lorentzian manifold (V^{n+1}, g) we have $(T_pV, g_p) \cong \mathbb{M}^{n+1}$. In particular, the Minkowski space-time is the arena where special relativity takes place.

We will now endow our Lorentzian manifolds with further structure than the minimal one imposed above. In particular, **we will always consider *time-orientable* Lorentzian manifolds, which we shall also refer to as *space-times***.

Definition 1.1.4. *(O'Neill 1983, Page 145) Let (V, g) be a Lorentzian manifold. At each point $p \in V$, in T_pV we have two null-cones. A choice of one of these null-cones is a time-orientation for T_pV. A smooth function τ on V which assigns to each $p \in V$ a null-cone in T_pV is said to be a time-orientation for V. We say (V, g) is time-orientable if it admits such a time-orientation function.*

It is straightforward to see that a Lorentzian manifold is time-orientable if and only if it admits a (global) time-like vector field (see, for instance, O'Neill (ibid., Lemma 32, Chapter 5).) Although in time-orientable Lorentzian manifolds there is a consistent way to distinguish past from future, these are still quite general structures which may inherit some exotic (maybe *undesirable*) properties. For instance, any compact Lorentzian manifold admits a closed time-like curve (see, for instance, O'Neill (ibid., Lemma 10, Chapter 14)). Since, within physics, causal paths represent the history of actual particles, this property is typically deemed as *pathological* allowing for potential travels to the past, and therefore excluded. Such exclusion is made by appealing to topological properties which guarantee a *good* causal structure on our space-time. Let us therefore introduce the relevant concepts.

Let (V, g) be a (time-orientable) Lorentzian manifold and $p, q \in V$.[4] Then, we will write:

1. $p \ll q$ if there is a future-pointing time-like curve in V from p to q;

[4] From now on, the time-orientability hypothesis will be implicitly assumed.

2. $p < q$ if there is a future-pointing causal curve in V from p to q;

3. $p \leqslant q$ if either $p < q$ or $p = q$;

4. Given a subset $A \subset V$, we define the *chronological* future $\mathcal{I}^+(A)$ and past $\mathcal{I}^-(A)$ of A by

$$\mathcal{I}^+(A) \doteq \{q \in V \ : \ \exists\, p \in A \text{ with } p \ll q\},$$
$$\mathcal{I}^-(A) \doteq \{q \in V \ : \ \exists\, p \in A \text{ with } q \ll p\},$$

and the *causal* future $\mathcal{J}^+(A)$ and past $\mathcal{J}^-(A)$ of A by

$$\mathcal{J}^+(A) \doteq \{q \in V \ : \ \exists\, p \in A \text{ with } p \leqslant q\},$$
$$\mathcal{J}^-(A) \doteq \{q \in V \ : \ \exists\, p \in A \text{ with } q \leqslant p\}.$$

There are several immediate consequences of these definitions, such as the fact the $\ll$ is always an open relation, implying that $\mathcal{I}^+(A)$ is always open, and also some subtleties, such as the fact that $\mathcal{J}^+(A)$ is not always closed (for a simple counter example, see O'Neill (ibid., Example 4, Chapter 14)). Nevertheless, since we shall only *use* this language to introduce relevant concepts and results, we will not be concerned with such subtleties and refer the interested reader to standard references, such as O'Neill (ibid.) or Hawking and Ellis (1973). Let us now introduce the following causality condition, which is related to our previous discussion.

Definition 1.1.5. *Let* (V, g) *be a Lorentzian manifold. We will say that* the strong causality condition *holds at* $p \in V$ *if for any given neighbourhood* $\mathcal{U}$ *of* p *there is a neighbourhood* $\mathcal{V} \subset \mathcal{U}$ *of* p *such that every causal curve with endpoints in* $\mathcal{V}$ *is entirely contained in* $\mathcal{U}$.

The above causality condition is basically tailored to exclude the possibility of *almost closed* causal-curves, since it implies that causal curves which leave a fixed neighbourhood of $p \in V$ cannot return to arbitrarily close to p. Again, deleting appropriate subsets of simple Lorentz manifolds can be shown to create a Lorentzian manifold without closed causal curves but with causal curves which are almost closed, and we intend to avoid this. In fact, it can be seen that if the strong causality condition holds in a compact subset K of a space-time (V, g), then *future-inextendible* causal curves in K eventually leave K and never return to it (O'Neill 1983, Lemma 13, Chapter 14).

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