Dynamics of Circle Mappings

Edson de Faria
Pablo Guarino

impa

33º Colóquio Brasileiro de Matemática

Dynamics of Circle Mappings

Dynamics of Circle Mappings
Primeira impressão, julho de 2021

Publicado no Brasil / Published in Brazil.

ISBN 978-65-89124-27-6
MSC (2020) Primary: 37E10, Secondary: 37C40, 37E20, 37F25, 37F31, 37F50

Coordenação Geral Carolina Araujo

Produção Books in Bytes **Capa** Izabella Freitas & Jack Salvador

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ

www.impa.br
editora@impa.br

Preface

One-dimensional Dynamics is a rich and beautiful subject, and the most authoritative work entirely dedicated to it still is, unquestionably, the book written by de Melo and van Strien [1993]. Thus the reader may ask: why bother writing another book about this subject?

It is a fair question. The main reason is that much has happened since 1993: more than half of the present book's contents deals with recent developments in the area. Moreover, rather than aiming at being comprehensive, our book delves deeper into a specific topic in One-dimensional Dynamics, namely, the dynamics of invertible circle maps. Let us say a few words explaining how this topic fits into the general framework of the modern theory of dynamical systems.

One of the major general goals in the area of Dynamical Systems is to solve the *smooth classification problem*: given two smooth dynamical systems which are topologically equivalent, when are they smoothly equivalent? In somewhat vague terms, this problem is tantamount to understanding the fine-scale geometric properties of such systems.

In such general setting, and particularly in higher dimensions, the above classification problem seems rather daunting (perhaps even hopeless). Hence one should first attempt to understand low-dimensional systems. At least at an intuitive level, the problem should be much simpler for one-dimensional systems; after all, in dimension one the linear order structure and "lack of ambient space" should impose severe restrictions on the possible geometries of such systems, thereby facilitating their smooth classification. However, even here the problem turns out to be rather subtle. A basic distinction that must be made in the one-dimensional context is between *invertible* dynamics – to wit, homeomorphisms of the circle – and *non-invertible* dynamics, such as the dynamics of unimodal or multimodal maps of the interval (or the circle).

In this book – written for a series of lectures delivered by both authors at the *33rd Brazilian Mathematics Colloquium* – we deal with *invertible* dynamical systems on the circle, concentrating on two major classes: global diffeomorphisms and smooth homeomorphisms with critical points. In the case of smooth diffeomorphisms of the circle, deep results have been obtained from the mid to late seventies onwards, starting with M. Herman's thesis and culminating with the work of J.-C. Yoccoz, with important contributions by Y. Katznelson and D. Ornstein, among others. After describing those results, we will focus on the case of smooth homeomorphisms with critical points, a topic to which both authors have dedicated several years of research. In this context, the notions of *renormalization*, *rigidity* and *universality* play a decisive role, and have been widely studied in the last thirty years.

The material of this book is divided into four parts. In the first part we study rigid rotations and then circle homeomorphisms, introducing the notion of *rotation number*, a dynamical invariant already considered by Poincaré at the end of the nineteen century. We also describe some connections between dynamical properties of the rotation number with the theory of continued fractions. In the second part we study circle diffeomorphisms, presenting some classical results due to Denjoy and discussing some of the main ideas in the Arnold-Herman-Yoccoz theory. We present the subject by developing it from its basic principles in a self-contained way. In particular, together, these two initial parts can be used on a first graduate-level course on one-dimensional dynamics. The book contains almost 100 exercises, varying widely in their level of difficulty; these should help the students enhance their understanding of the subject.

The third part of this book introduces smooth homeomorphisms of the circle with a finite number of critical points, an important and active topic in the area of one-dimensional dynamics. The fourth and last part of this book is devoted to Renormalization Theory, focusing on the analysis of the fine geometric structure of orbits of multicritical circle maps, as well as on certain complex-analytic aspects of the subject. We will describe in these final chapters several important results by K. Khanin, M. Martens, C. McMullen, W. de Melo, D. Sullivan, A. Teplinsky and M. Yampolsky among others. We would like to remark that, since these ideas are quite deep, the narrative in this final part is by necessity very sketchy.

Throughout the book, we provide, for the most part, complete proofs of several fundamental results in circle dynamics, such as the Poincaré classification, Denjoy's classical results and constructions, Arnold's conjugacy theorem for analytic circle diffeomorphisms with Diophantine rotation number (we also describe his counterexamples to linearizability), Yoccoz's theorem on minimality of multicritical circle maps, the *real bounds*, quasisymmetric rigidity, the fact that exponential convergence of renormalization implies smooth rigidity, Lipschitz continuity of the renormalization operator (for maps with a single critical point) and the *complex bounds*. We also survey, skipping many details, the proof of the exponential convergence of renormalization for critical circle maps, both in the analytic and the smooth case. The book closes with an appendix describing some aspects of the ergodic theory of continued fractions.

The present book is primarily aimed at graduate students and young researchers working in Dynamical Systems, but we hope it will have something to offer to other mathematicians interested in the subject. As prerequisites, it assumes that the reader is familiar with the contents of a standard graduate course in Real Analysis (including Metric Spaces, Measure Theory and basic Functional Analysis) in addition to some notions of Ergodic Theory and Dynamical Systems. In chapters 4, 11, 13 and 14, basic knowledge of Complex Analysis is needed as well.

Due to limitations of time and space, many interesting topics of circle dynamics have been left out of this book. These include interval exchange transformations, maps with break points, mode locking universality, dynamics of endomorphisms (including the notion of rotation set), thermodynamic formalism, invariant distributions, random dynamical systems and groups acting on the circle, among others.

In recent years, we have benefited from conversations with many friends and colleagues, among them Marco Martens, Sebastian van Strien, Dennis Sullivan, Charles Tresser, Björn Winckler, Misha Yampolsky and most notably Welington de Melo. Several parts of this book have been inspired by these interactions.

We would like to thank the organizers of the 33rd Brazilian Mathematics Colloquium for the opportunity to present this course. Special thanks go to Paulo Ney de Souza for his extremely professional editorial help. Readers are encouraged to send comments and suggestions as well as corrections to our email addresses.

Edson de Faria (`edson@ime.usp.br`)
Pablo Guarino (`pablo_guarino@id.uff.br`)

The first author has been supported by "Projeto Temático Dinâmica em Baixas Dimensões" FAPESP Grant 2016/25053-8, while the second author has been supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) grant 23038.009189/2013-05.

Contents

Part I

Basic Theory

Títulos Publicados — 33º Colóquio Brasileiro de Matemática

Geometria Lipschitz das singularidades – *Lev Birbrair e Edvalter Sena*

Combinatória – *Fábio Botler, Maurício Collares, Taísa Martins, Walner Mendonça, Rob Morris e Guilherme Mota*

Códigos Geométricos – *Gilberto Brito de Almeida Filho e Saeed Tafazolian*

Topologia e geometria de 3-variedades – *André Salles de Carvalho e Rafał Marian Siejakowski*

Ciência de Dados: Algoritmos e Aplicações – *Luerbio Faria, Fabiano de Souza Oliveira, Paulo Eustáquio Duarte Pinto e Jayme Luiz Szwarcfiter*

Discovering Euclidean Phenomena in Poncelet Families – *Ronaldo A. Garcia e Dan S. Reznik*

Introdução à geometria e topologia dos sistemas dinâmicos em superfícies e além – *Víctor León e Bruno Scárdua*

Equações diferenciais e modelos epidemiológicos – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*

Differential Equation Models in Epidemiology – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*

A friendly invitation to Fourier analysis on polytopes – *Sinai Robins*

PI-álgebras: uma introdução à PI-teoria – *Rafael Bezerra dos Santos e Ana Cristina Vieira*

First steps into Model Order Reduction – *Alessandro Alla*

The Einstein Constraint Equations – *Rodrigo Avalos e Jorge H. Lira*

Dynamics of Circle Mappings – *Edson de Faria e Pablo Guarino*

Statistical model selection for stochastic systems – *Antonio Galves, Florencia Leonardi e Guilherme Ost*

Transfer Operators in Hyperbolic Dynamics – *Mark F. Demers, Niloofar Kiamari e Carlangelo Liverani*

A Course in Hodge Theory Periods of Algebraic Cycles – *Hossein Movasati e Roberto Villaflor Loyola*

A dynamical system approach for Lane–Emden type problems – *Liliane Maia, Gabrielle Nornberg e Filomena Pacella*

Visualizing Thurston's Geometries – *Tiago Novello, Vinícius da Silva e Luiz Velho*

Scaling Problems, Algorithms and Applications to Computer Science and Statistics – *Rafael Oliveira e Akshay Ramachandran*

An Introduction to Characteristic Classes – *Jean-Paul Brasselet*

impa

Instituto de
Matemática
Pura e Aplicada

ISBN 978-65-89124-27-6

9 786589 124276