

# Dynamics of Circle Mappings

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33<sup>o</sup> Colóquio  
Brasileiro de  
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# **Dynamics of Circle Mappings**

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# Preface

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One-dimensional Dynamics is a rich and beautiful subject, and the most authoritative work entirely dedicated to it still is, unquestionably, the book written by de Melo and van Strien [1993]. Thus the reader may ask: why bother writing another book about this subject?

It is a fair question. The main reason is that much has happened since 1993: more than half of the present book's contents deals with recent developments in the area. Moreover, rather than aiming at being comprehensive, our book delves deeper into a specific topic in One-dimensional Dynamics, namely, the dynamics of invertible circle maps. Let us say a few words explaining how this topic fits into the general framework of the modern theory of dynamical systems.

One of the major general goals in the area of Dynamical Systems is to solve the *smooth classification problem*: given two smooth dynamical systems which are topologically equivalent, when are they smoothly equivalent? In somewhat vague terms, this problem is tantamount to understanding the fine-scale geometric properties of such systems.

In such general setting, and particularly in higher dimensions, the above classification problem seems rather daunting (perhaps even hopeless). Hence one should first attempt to understand low-dimensional systems. At least at an intuitive level, the problem should be much simpler for one-dimensional systems; after all, in dimension one the linear order structure and “lack of ambient space” should impose severe restrictions on the possible geometries of such systems, thereby facilitating their smooth classification. However, even here the problem turns out to be rather subtle. A basic distinction that must be made in the one-dimensional context is between *invertible* dynamics – to wit, homeomorphisms of the circle – and *non-invertible* dynamics, such as the dynamics of unimodal or multimodal maps of the interval (or the circle).

In this book – written for a series of lectures delivered by both authors at the *33rd Brazilian Mathematics Colloquium* – we deal with *invertible* dynamical systems on the circle, concentrating on two major classes: global diffeomorphisms and smooth homeomorphisms with critical points. In the case of smooth diffeomorphisms of the circle, deep results have been obtained from the mid to late seventies onwards, starting with M. Herman’s thesis and culminating with the work of J.-C. Yoccoz, with important contributions by Y. Katznelson and D. Ornstein, among others. After describing those results, we will focus on the case of smooth homeomorphisms with critical points, a topic to which both authors have dedicated several years of research. In this context, the notions of *renormalization*, *rigidity* and *universality* play a decisive role, and have been widely studied in the last thirty years.

The material of this book is divided into four parts. In the first part we study rigid rotations and then circle homeomorphisms, introducing the notion of *rotation number*, a dynamical invariant already considered by Poincaré at the end of the nineteenth century. We also describe some connections between dynamical properties of the rotation number with the theory of continued fractions. In the second part we study circle diffeomorphisms, presenting some classical results due to Denjoy and discussing some of the main ideas in the Arnold-Herman-Yoccoz theory. We present the subject by developing it from its basic principles in a self-contained way. In particular, together, these two initial parts can be used on a first graduate-level course on one-dimensional dynamics. The book contains almost 100 exercises, varying widely in their level of difficulty; these should help the students enhance their understanding of the subject.

The third part of this book introduces smooth homeomorphisms of the circle with a finite number of critical points, an important and active topic in the area of one-dimensional dynamics. The fourth and last part of this book is devoted to Renormalization Theory, focusing on the analysis of the fine geometric structure of orbits of multicritical circle maps, as well as on certain complex-analytic aspects of the subject. We will describe in these final chapters several important results by K. Khanin, M. Martens, C. McMullen, W. de Melo, D. Sullivan, A. Teplinsky and M. Yampolsky among others. We would like to remark that, since these ideas are quite deep, the narrative in this final part is by necessity very sketchy.

Throughout the book, we provide, for the most part, complete proofs of several fundamental results in circle dynamics, such as the Poincaré classification, Denjoy’s classical results and constructions, Arnold’s conjugacy theorem for analytic circle diffeomorphisms with Diophantine rotation number (we also describe his counterexamples to linearizability), Yoccoz’s theorem on minimality of multicriti-

cal circle maps, the *real bounds*, quasimetric rigidity, the fact that exponential convergence of renormalization implies smooth rigidity, Lipschitz continuity of the renormalization operator (for maps with a single critical point) and the *complex bounds*. We also survey, skipping many details, the proof of the exponential convergence of renormalization for critical circle maps, both in the analytic and the smooth case. The book closes with an appendix describing some aspects of the ergodic theory of continued fractions.

The present book is primarily aimed at graduate students and young researchers working in Dynamical Systems, but we hope it will have something to offer to other mathematicians interested in the subject. As prerequisites, it assumes that the reader is familiar with the contents of a standard graduate course in Real Analysis (including Metric Spaces, Measure Theory and basic Functional Analysis) in addition to some notions of Ergodic Theory and Dynamical Systems. In chapters 4, 11, 13 and 14, basic knowledge of Complex Analysis is needed as well.

Due to limitations of time and space, many interesting topics of circle dynamics have been left out of this book. These include interval exchange transformations, maps with break points, mode locking universality, dynamics of endomorphisms (including the notion of rotation set), thermodynamic formalism, invariant distributions, random dynamical systems and groups acting on the circle, among others.

In recent years, we have benefited from conversations with many friends and colleagues, among them Marco Martens, Sebastian van Strien, Dennis Sullivan, Charles Tresser, Björn Winckler, Misha Yampolsky and most notably Wellington de Melo. Several parts of this book have been inspired by these interactions.

We would like to thank the organizers of the 33rd Brazilian Mathematics Colloquium for the opportunity to present this course. Special thanks go to Paulo Ney de Souza for his extremely professional editorial help. Readers are encouraged to send comments and suggestions as well as corrections to our email addresses.

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# Contents

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<b>Preface</b>	<b>i</b>
<b>I Basic Theory</b>	<b>1</b>
<b>1 Rotations</b>	<b>2</b>
1.1 Topology and combinatorics of rotations . . . . .	2
1.1.1 A dichotomy . . . . .	3
1.1.2 Sequence of closest returns . . . . .	5
1.2 Rotations and continued fractions . . . . .	7
1.2.1 Basic theory of continued fractions . . . . .	7
1.2.2 Best approximations . . . . .	12
1.3 Weyl's equidistribution theorem . . . . .	14
1.3.1 Equidistribution . . . . .	14
1.3.2 A simple application . . . . .	17
Exercises . . . . .	18
<b>2 Homeomorphisms of the Circle</b>	<b>21</b>
2.1 Translation and rotation numbers . . . . .	22
2.1.1 The classical definition . . . . .	22
2.1.2 The order definition . . . . .	23
2.1.3 The measure-theoretic definition . . . . .	24
2.1.4 Properties of the rotation number . . . . .	25

2.2	Topological dynamics of homeomorphisms . . . . .	27
2.2.1	Rational rotation number . . . . .	27
2.2.2	Irrational rotation number . . . . .	29
2.3	Invariant measures and semi-conjugacies . . . . .	31
	Exercises . . . . .	34

## **II Diffeomorphisms 38**

### **3 Diffeomorphisms: Denjoy Theory 39**

3.1	The naive distortion lemma . . . . .	39
3.2	Denjoy's theorem . . . . .	41
3.2.1	The $C^2$ version . . . . .	41
3.2.2	The bounded variation version . . . . .	43
3.3	Denjoy's examples . . . . .	45
3.3.1	The basic construction . . . . .	46
3.3.2	Moduli of continuity . . . . .	52
3.3.3	Further results . . . . .	55
3.4	Ergodic properties . . . . .	58
3.4.1	Ergodicity with respect to Lebesgue measure . . . . .	58
3.4.2	Zero Lyapunov exponents . . . . .	61
	Exercises . . . . .	62

### **4 Smooth Conjugacies to Rotations 64**

4.1	Herman's invariants . . . . .	65
4.2	Small denominators: Arnold's theorem . . . . .	68
4.2.1	The linearized equation . . . . .	71
4.2.2	Non-linear estimates . . . . .	75
4.2.3	Proof of Arnold's theorem . . . . .	78
4.3	Counterexamples to linearizability . . . . .	81
4.3.1	One-parameter families . . . . .	82
4.3.2	Residual sets of non-linearizable parameters . . . . .	85
4.3.3	Singular measures and conjugacies . . . . .	86
4.4	Further local theory: the Brjuno condition . . . . .	91
4.5	Global theory: Herman–Yoccoz results and beyond . . . . .	93
	Exercises . . . . .	94



<b>III</b>	<b>Multicritical Circle Maps</b>	<b>97</b>
<b>5</b>	<b>Cross-ratios and Distortion Tools</b>	<b>98</b>
5.1	Cross-ratios . . . . .	98
5.2	The Schwarzian . . . . .	99
5.2.1	Definition . . . . .	100
5.2.2	Koebe's non-linearity principle . . . . .	101
5.2.3	The minimum principle . . . . .	103
5.3	Distortion and cross-ratio distortion . . . . .	104
5.3.1	Koebe's distortion principle . . . . .	104
5.3.2	Distortion and the Schwarzian . . . . .	104
5.3.3	Behavior near critical points . . . . .	107
5.4	The Cross-Ratio Inequality . . . . .	110
5.5	A cancellation lemma . . . . .	111
	Exercises . . . . .	116
<b>6</b>	<b>Topological Classification and the Real Bounds</b>	<b>119</b>
6.1	Definition and examples of multicritical circle maps . . . . .	120
6.1.1	Blaschke products . . . . .	120
6.1.2	The Arnold family . . . . .	122
6.2	Topological classification . . . . .	124
6.2.1	Dynamically symmetric intervals . . . . .	125
6.2.2	Proof of Yoccoz's theorem . . . . .	130
6.3	Real a priori bounds . . . . .	130
6.3.1	Dynamical partitions . . . . .	131
6.3.2	The real bounds . . . . .	133
6.3.3	On the notion of comparability . . . . .	139
6.4	First consequences . . . . .	140
6.4.1	$C^1$ bounds . . . . .	140
6.4.2	Sums of polar ratios . . . . .	143
6.5	A negative Schwarzian property . . . . .	145
6.6	Beau bounds . . . . .	147
	Exercises . . . . .	151
<b>7</b>	<b>Quasisymmetric Rigidity</b>	<b>152</b>
7.1	Quasisymmetry and fine grids . . . . .	152
7.1.1	A criterion for quasisymmetry . . . . .	154
7.1.2	A criterion for smoothness . . . . .	157

7.2	Quasisymmetric conjugacies . . . . .	160
7.3	Almost parabolic maps . . . . .	161
7.3.1	Yoccoz’s inequality . . . . .	162
7.3.2	Balanced decompositions . . . . .	166
7.4	Quasisymmetric rigidity . . . . .	169
7.4.1	More on the geometry of dynamical partitions . . . . .	170
7.4.2	Building a suitable fine grid . . . . .	174
7.4.3	The punchline . . . . .	181
	Exercises . . . . .	182
<b>8</b>	<b>Ergodic Aspects</b>	<b>185</b>
8.1	The integrability of $\log Df$ . . . . .	185
8.2	No invariant $\sigma$ -finite measures . . . . .	189
8.2.1	The Katznelson criterion . . . . .	190
8.3	Lyapunov exponents . . . . .	194
8.3.1	The Collet–Eckmann condition . . . . .	194
8.3.2	The key step . . . . .	194
8.4	Hausdorff dimension . . . . .	200
	Exercises . . . . .	201
<b>9</b>	<b>Orbit Flexibility</b>	<b>202</b>
9.1	Geometric equivalence of orbits . . . . .	203
9.1.1	Orbit-flexibility . . . . .	203
9.1.2	Statement for unicritical maps . . . . .	204
9.1.3	Statements for multicritical maps . . . . .	205
9.1.4	Centralizers . . . . .	206
9.1.5	Unbounded geometry . . . . .	208
9.2	Renormalization trails and ancestors . . . . .	209
9.3	The skew product . . . . .	212
9.3.1	The fiber maps . . . . .	212
9.3.2	The skew product . . . . .	212
9.4	Proof of Theorem 9.6 . . . . .	214
9.5	Even-type rotation numbers . . . . .	215
9.6	Proofs of Theorems 9.1 and 9.2 . . . . .	216
	Exercises . . . . .	220

<b>IV Renormalization Theory</b>	<b>221</b>
<b>10 Smooth Rigidity and Renormalization</b>	<b>222</b>
10.1 Smooth rigidity . . . . .	223
10.2 Renormalization of commuting pairs . . . . .	227
10.3 A fundamental principle . . . . .	233
10.3.1 Main theorem . . . . .	233
10.3.2 Comparing orbits of two almost parabolic maps . . . . .	235
10.3.3 Proof of Theorem 10.4 . . . . .	237
10.4 The $C^m$ -Approximation Lemma . . . . .	245
10.5 Counterexamples to $C^{1+\alpha}$ rigidity . . . . .	248
10.5.1 Saddle-node surgery . . . . .	249
10.5.2 The counterexamples . . . . .	252
Exercises . . . . .	253
<b>11 Quasiconformal deformations</b>	<b>254</b>
11.1 Quasiconformal homeomorphisms . . . . .	255
11.1.1 The geometric definition . . . . .	255
11.1.2 The analytic definition . . . . .	256
11.1.3 Measurable Riemann mapping theorem . . . . .	257
11.2 A simple dynamical application . . . . .	259
11.3 Holomorphic approximation lemma . . . . .	260
Exercises . . . . .	265
<b>12 Lipschitz estimates for renormalization</b>	<b>268</b>
12.1 Standard families . . . . .	270
12.1.1 Glueing procedure and translations . . . . .	270
12.1.2 Standard families of commuting pairs . . . . .	272
12.1.3 Renormalization of standard families . . . . .	274
12.2 Orbit Deformations . . . . .	278
12.3 Composition . . . . .	293
12.4 Order . . . . .	300
12.5 Synchronization . . . . .	302
12.6 Lipschitz Estimate . . . . .	305
Exercises . . . . .	306
<b>13 Exponential convergence: the smooth case</b>	<b>307</b>
13.1 The shadowing property . . . . .	308
13.1.1 Extended lifts of critical circle maps . . . . .	310

13.1.2	Almost Schwarz inclusion . . . . .	314
13.1.3	A bidimensional glueing procedure . . . . .	317
13.1.4	Main perturbation . . . . .	325
13.1.5	The shadowing sequence . . . . .	334
13.2	Bounding the $C^{r-1}$ metric . . . . .	336
13.3	Proof of the exponential convergence . . . . .	337
	Exercises . . . . .	338
<b>14</b>	<b>Renormalization: Holomorphic methods</b>	<b>339</b>
14.1	Sullivan's program . . . . .	339
14.2	Holomorphic commuting pairs . . . . .	344
14.3	Pull-back argument . . . . .	346
14.4	Complex bounds . . . . .	347
14.5	McMullen's dynamic inflexibility theorem . . . . .	354
14.6	Proof of exponential convergence of renormalizations . . . . .	356
14.7	Hyperbolicity of renormalization . . . . .	358
<b>A</b>	<b>Ergodic Theory of Continued Fractions</b>	<b>359</b>
	Exercises . . . . .	367
	<b>Bibliography</b>	<b>369</b>
	<b>Index</b>	<b>380</b>

**Part I**

**Basic Theory**

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