Transfer operators in Hyperbolic Dynamics
An introduction

Mark F. Demers
Niloofar Kiamari
Carlangelo Liverani

impa

33º Colóquio Brasileiro de Matemática

Transfer operators in Hyperbolic Dynamics
An introduction

Transfer operators in Hyperbolic Dynamics An introduction
Primeira impressão, julho de 2021

Publicado no Brasil / Published in Brazil.

ISBN 978-65-89124-26-9
MSC (2020) Primary: 37C30, Secondary: 37D20, 37C05, 37E35, 47B38, 37A25

Coordenação Geral Carolina Araujo

Produção Books in Bytes **Capa** Izabella Freitas & Jack Salvador

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ

www.impa.br
editora@impa.br

Preface

This text is a result of the notes written for several Schools. It started with a series of lectures, *Probability and uniformly hyperbolic systems*, given by Carlangelo Liverani in Coimbra in 2008 and the lectures delivered by Mark Demers and Carlangelo Liverani at the *International Conference on Statistical Properties of Non-equilibrium Dynamical Systems*, SUSTC, Shenzhen, July 27 – August 2, 2016. It was then modified and used for the lectures *An introduction to the statistical properties of hyperbolic dynamical systems*, delivered by Carlangelo Liverani at the TMU-ICTP School, Tehran, May 5 – 10, 2018. It has finally reached its present state for the lectures by Carlangelo Liverani at the 33º Colóquio Brasileiro de Matemática.

Our aim is not to make a review of the field, but rather to introduce the reader to some basic modern techniques used to study the statistical properties of *chaotic* systems. Here by *chaotic* we mean uniformly hyperbolic systems. That is, systems that display a strong uniform sensitivity with respect to initial conditions. We will stress in particular the so-called *functional approach*, but we will also provide a simple introduction to the use of *standard pairs* and *Hilbert metrics*, and discuss some of the relations among these tools.

The goal is to provide the reader with a quick introduction to the literature. On the one hand we describe in detail the main techniques when applied to the simplest cases, providing full proofs for the essential general facts of the theory. On the other hand we try to flesh out the fundamental ideas necessary to understand the current literature, while avoiding the most technical details.

This note is a partial update with respect to the small review Liverani (2003).

For a much more in depth and technical discussion of transfer operators see Baladi (2000, 2018).

Our main focus, the functional approach, has its origin in the study of the Koopman operator Koopman (1931) (acting on L^2) starting, at least, with the von Neumann mean ergodic theorem von Neumann (1932) and further developed by the Russian school, see Cornfeld, Fomin, and Sinai (1982). An important development of this point of view occurred with the study of the transfer operator in symbolic dynamics by Sinai (1968, 1972b), Ruelle (1976a, 1978) and Bowen (1970, 2008).

Next, the functional approach developed further thanks to the work of Lasota and Yorke (1973), Ruelle (1976b), Keller (1979), Hofbauer and Keller (1982) and, more recently, Fried (1986), Rugh (1992, 1996), and Kitaev (1999), just to mention a few. This has eventually led to the current theory, which has assumed its present form starting with Blank, Keller, and Liverani (2002).

The basic idea is to study directly the spectrum of the Ruelle transfer operator without coding the system (even though the theory can be applied also to the transfer operator of a system after inducing). In order to do so, it is necessary to consider the action of the transfer operator on an appropriate Banach (or Hilbert) space or, more generally, in an appropriate topology. The non trivial part of the theory rests in the identification of the appropriate topological spaces which, to be effective, must reflect the geometric features of the dynamics.

In this note we will discuss only uniformly hyperbolic systems, yet the techniques presented here are relevant also in the non uniformly hyperbolic case, although they must be supplemented with essential new ideas such as Young towers, started by Young (1998); coupling, as introduced by Dolgopyat (2000) and Young (1999); and Operator Renewal Theory, whose development is due to Sarig (1999). In fact, it may be interesting to combine different techniques in order to develop a more effective theory: examples of attempts in this direction are De Simoi and Liverani (2016) and Maume-Deschamps (2001).

Another of our goals is to explain which properties the above mentioned Banach spaces must enjoy and to provide a guide for how to construct and adapt them to the peculiarities of the systems at hand. Also, we will briefly discuss the idea of coupling in an especially simple case, but we will not provide any details regarding Young towers or Operator Renewal Theory. More generally, we will not discuss non-uniform hyperbolicity nor general partial hyperbolicity (for the latter we refer to the book Bonatti, Díaz, and Viana (2005)).

The plan of the exposition is as follows: we start, in Chapter 1, discussing the simplest possible case, smooth expanding maps of the circle. This allows us to illustrate, in the simplest possible setting, the power of the functional approach and the type of results that can be obtained once such machinery is in place. In particular, we will show how important properties of the system such as exponential decay of correlations, the Central Limit Theorem, Large deviation results, stability and linear response easily follow from the spectral properties of the transfer operator.

In Chapter 2, we will discuss the case of attractors, where the need to consider spaces of distributions first becomes apparent.

In Chapter 3 we develop the theory for the case of toral automorphisms. This may seem a bit silly as toral automorphisms can be studied directly using Fourier series. Yet, this will allow us to illustrate, in the most elementary manner, the main ideas of the theory, including anisotropic Banach spaces and coupling.

Then, in Chapter 4, we collect all the ideas previously illustrated and extend them to study general uniformly hyperbolic maps. This gives a precise taste of what the full theory looks like for uniformly hyperbolic systems.

Next, we discuss non-singular flows. By non-singular we mean that the vector field generating the flow has no zeros. This implies that a Lyapunov exponent (the one in the flow direction) is necessarily zero. Hence, this is one of the simplest possible partially hyperbolic systems. The other simple classes of partially hyperbolic systems are skew-products and group extensions. Some of these can be treated with similar techniques, but we will not discuss them explicitly in this note.

We will restrict ourselves to the case of *contact* flows. Although much of the present theory can be applied, with few changes, to more general hyperbolic flows, the contact flow case is the simplest example and hence well suited to an introductory discussion.

There are three main steps in adapting the analysis of the discrete time transfer operator for hyperbolic maps to the semi-group of continuous time transfer operators for hyperbolic flows:

1. Adapt Banach spaces used for hyperbolic maps to the setting of hyperbolic flows: the presence of the neutral flow direction makes this a nontrivial change.

2. Contrary to the discrete-time case, we do not prove the quasi-compactness of the transfer operator for the time-one map of the flow, but rather for the generator of the semi-group of transfer operators for the flow; this involves the use of the resolvent to 'integrate out' the neutral direction.

3. The use of the contact form to estimate an oscillatory integral and derive a spectral gap for the generator of the semi-group and an estimate for the resolvent close to the imaginary axis (the Dolgopyat-type estimate).

It then follows from some general considerations that a spectral gap for the generator of the semi-group implies exponential decay of correlations for the flow. This approach is detailed in Chapter 5.

At last Chapter 6 discusses the extension of these ideas to hyperbolic billiards. Note that hyperbolic billiards have serious discontinuities, hence albeit the overall strategy is the same as in the smooth case, there are crucial technical problems to overcome, problems that delayed the extension of the theory to this type of system for almost 20 years.

The notes also include several appendices. These are aimed at providing the reader with some basic knowledge that, while necessary to fully understand the main text, is not necessarily common knowledge.

Appendix A contains some very basic facts concerning functional analysis. These are normally covered in a graduate functional analysis course, but, just in case the reader was distracted, here we provide the minimum necessary to understand the main text.

Appendix B is devoted to a full exposition of the Hennion–Neussbaum theory. Such a theory underlies much of the current approach, yet it is impossible to find a full exposition of such results that has as prerequisite only the content of a standard first functional analysis course. We think that it is better to have full control of the main instruments used in the field, hence we attempt to fill this expository gap.

Appendix C presents a simplified version of the perturbative theory developed in Keller and Liverani (1999) and Gouëzel and Liverani (2006). Although not necessary to understand the main text, this theory is by now a standard tool to study the dependence of the statistical properties of a system on a parameter or external influences. Hence, it is natural to add it for completeness.

Appendix D contains the basics of projective cones and Hilbert metrics. Part of this material can also be found in other books (e.g., Viana (1997)) but we add it for completeness. Also we emphasize the connection with the Banach space approach, which is not common knowledge.

Appendix E contains hints to the solutions of the problems in the text. We strongly recommend that the readers look at this appendix only as a last resort and only after some hard thinking in order to find a solution.

To conclude we would like to thank all the people that provided us with helpful suggestions related to this text. They are too many to name but, at least, we must mention Viviane Baladi, Oliver Butterley, Jacopo de Simoi, Dmitry Dolgopyat and Sébastien Gouëzel.

Contents

1 Expanding maps

We start by discussing *smooth expanding maps*. By a smooth expanding map we mean a map $f \in \mathcal{C}^r(\mathbb{T}, \mathbb{T})$,[1] $r \geqslant 2$, such that $\inf_x |f'(x)| \geqslant \lambda_\star > 1$. Clearly $(f, \mathbb{T})$ is a topological, actually differentiable, dynamical system. Our first goal is to view it as a measurable dynamical system, hence we need to select an invariant probability measure.

1.1 Invariant measures

Deterministic systems often have a lot of invariant measures. In particular, to any periodic orbit is associated an invariant measure (the average along the orbit). Given such plentiful possibilities, we need a criteria to select relevant invariant measures. A common choice is to consider measures that can be obtained by pushing forward a measure absolutely continuous with respect to Lebesgue.

More precisely, let $d\mu = h(x)dx$, $h \in L^1(\mathbb{T}, \text{Leb})$ and define,[2] for all $\varphi \in$

[1] By $\mathbb{T}$ we mean the one dimensional torus $\mathbb{R}/\mathbb{Q}$. While $\mathcal{C}^r$, as usual, denotes the set of functions r times differentiable with continuous derivatives.

[2] Obviously, Leb stands for the Lebesgue measure.

Títulos Publicados — 33º Colóquio Brasileiro de Matemática

Geometria Lipschitz das singularidades – *Lev Birbrair e Edvalter Sena*

Combinatória – *Fábio Botler, Maurício Collares, Taísa Martins, Walner Mendonça, Rob Morris e Guilherme Mota*

Códigos Geométricos – *Gilberto Brito de Almeida Filho e Saeed Tafazolian*

Topologia e geometria de 3-variedades – *André Salles de Carvalho e Rafał Marian Siejakowski*

Ciência de Dados: Algoritmos e Aplicações – *Luerbio Faria, Fabiano de Souza Oliveira, Paulo Eustáquio Duarte Pinto e Jayme Luiz Szwarcfiter*

Discovering Euclidean Phenomena in Poncelet Families – *Ronaldo A. Garcia e Dan S. Reznik*

Introdução à geometria e topologia dos sistemas dinâmicos em superfícies e além – *Víctor León e Bruno Scárdua*

Equações diferenciais e modelos epidemiológicos – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*

Differential Equation Models in Epidemiology – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*

A friendly invitation to Fourier analysis on polytopes – *Sinai Robins*

PI-álgebras: uma introdução à PI-teoria – *Rafael Bezerra dos Santos e Ana Cristina Vieira*

First steps into Model Order Reduction – *Alessandro Alla*

The Einstein Constraint Equations – *Rodrigo Avalos e Jorge H. Lira*

Dynamics of Circle Mappings – *Edson de Faria e Pablo Guarino*

Statistical model selection for stochastic systems – *Antonio Galves, Florencia Leonardi e Guilherme Ost*

Transfer Operators in Hyperbolic Dynamics – *Mark F. Demers, Niloofar Kiamari e Carlangelo Liverani*

A Course in Hodge Theory Periods of Algebraic Cycles – *Hossein Movasati e Roberto Villaflor Loyola*

A dynamical system approach for Lane–Emden type problems – *Liliane Maia, Gabrielle Nornberg e Filomena Pacella*

Visualizing Thurston's Geometries – *Tiago Novello, Vinícius da Silva e Luiz Velho*

Scaling Problems, Algorithms and Applications to Computer Science and Statistics – *Rafael Oliveira e Akshay Ramachandran*

An Introduction to Characteristic Classes – *Jean-Paul Brasselet* impa

Instituto de
Matemática
Pura e Aplicada

ISBN 978-65-89124-26-9

9 786589 124269