A course in Hodge Theory: Periods of algebraic cycles

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در آتش خویش چون دمی جوش کنم
خواهم که دمی ترا فراموش کنم
گیرم جانی که عقل بیهوش کند
در جام درآئی و ترا نوش کنم

If I boil in the fire of my existence for a while,
that is because I want to forget you for a while,
to get a new soul and put away my wisdom,
and then you become the wine of my glass.

Quatrain № 1215 by Rumi (Jalal al-Din Muhammad Balkhi) published in *Kulliyat-e Shams-e Tabrizi*, with translation by the first author, Hossein Movasati.

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Preface

The present book is a sequel to the first author's book "A Course in Hodge Theory: With Emphasis on Multiple Integrals". The first book focuses mainly on affine hypersurfaces and the study of the Hodge locus through the Fermat variety and uses techniques from singularity theory, such as Brieskorn modules and vanishing cycles. This book intends to tell us the Hodge theory of smooth projective varieties and their properties inside families. This is the study of Čech cohomology, hypercohomology, Algebraic de Rham cohomology, Gauss–Manin connection, infinitesimal variation of Hodge structures and Hodge loci. Despite this, in order to do concrete computations, we will be back to our favorite example of hypersurfaces. Both books intend to make Hodge theory as computational as possible, either by hand or by computer. Together with other classical books in Hodge theory such as Voisin's two volume books, they can be used in a Graduate course. It is mainly for students and researchers who want to study the Hodge conjecture in families. We assume a basic knowledge in both Algebraic Topology and Algebraic Geometry.

Hossein Movasati, Roberto Villaflor Loyola
May 2021, Rio de Janeiro, RJ, Brazil

1 Introduction

The first draft of the present book was the lecture notes of a second course in Hodge theory presented by the first author in 2015 to the second author. These notes were developed into the second author's Ph.D. thesis and the present text is the outcome of this collaboration.

1.1 Computational Hodge theory

In order to solve a mathematical problem one may generalize it until a solution comes out by itself and this method is, for instance, present in Grothendieck's philosophy. A completely different approach must be adopted if one has the feeling that the Hodge conjecture is wrong. Instead of generalizations, one has to study so many particular examples, and one has to compute so many well-known theoretical data, such that the counterexample comes out by itself. Once you are in the ocean without compass, all directions might lead you to a land. If such a counterexample is found then it would be like a Columbus' egg and there will be an explosion of other counterexamples. Even if the Hodge conjecture is true, the belief that it is false makes us to take a more computational approach, and at least this makes the Hodge theory accessible to broader class of mathematicians, and in particular those who love computational mathematics, either by hand or by computer. The first author's book Movasati (2021) is the first attempt in this direction. Since the main focus of this book is smooth hypersurfaces, we feel that one has to prepare the computational ground for arbitrary smooth projective varieties and the present text is the output of this attempt.

1.2 Some missing details

In an attempt to make a piece of mathematics more computational, one might think that all the credit belongs to theory makers and one does only hard exercises of these theories, and therefore, this kind of mathematics might be banned from publication in "prestigious journals". The amount of time and effort needed for this purpose is usually higher than if we wanted to contribute for producing more theories. One may also find some missing details in the work of theory makers which are as important as the main body of such theories. Here, we would like to highlight one example. In a personal communication (November 09, 2018) with P. Deligne, the first author posed the following question: " I got motivated to write this email after seeing your talk 'what do we mean by equal', and after getting the feeling that in the foundation of Hodge theory not every detail is explained. Let me explain this. For a smooth projective variety over $\mathbb{C}$ we know that there is a concrete canonical isomorphism between the usual de Rham cohomology by C^∞ forms and the algebraic de Rham cohomology. Therefore, all the concepts in the topological side, such as cup product, cohomology class of cycles etc. can be transported to the algebraic side and can be defined in a purely algebraic fashion. However, my impression is that nobody has verified concretely that the algebraic objects are the exact transportation of the corresponding topological objects. For instance, Grothendieck's definition of a class of an algebraic cycle must corresponds to the one defined by integration, but I do not see if it is written somewhere. In particular, when the algebraic cycle is singular, the only rigorous proof that I see is the resolution of singularities. Anyway, with my student Roberto Villaflor, we are trying to write a book containing the maximum details, however, we are stuck in this kind of issue. I would be grateful if you clarify this for us." The answer came few days later. "I have not thought about proving the compatibility between Grothendieck's definition of the class of a cycle and integration, because I never used the latter. I like to use cohomology, not homology, and I like definitions which are uniform across cohomology theories (motivic philosophy)...", (P. Deligne, personal communication November 13, 2018). As an idiom says "the devil is in the detail" and such missing details in the literature took more than 4 years of both authors.

1.3 The organization of the text

The emphasis of the first book Movasati (ibid.) was mainly on hypersurfaces and the study of the Hodge locus through the Fermat variety. This book intends to tell us the Hodge theory of smooth projective varieties and their properties inside families. This is the study of Čech cohomology, hypercohomology, Gauss–Manin connection, infinitesimal variation of Hodge structures, Hodge loci etc. Despite this, in order to do concrete computations, we will be back to our favorite example of hypersurfaces. A synopsis of each chapter is explained below.

In Chapter 2 we present a minimum amount of material so that the reader gets familiar with Čech cohomology. We need to represent elements of cohomologies with concrete data and we do this using an acyclic covering. In Chapter 3 we aim to define hypercohomology of a complex of sheaves relative to an acyclic covering, and hence, we describe elements of a hypercohomology with concrete data. We discuss quasi-isomorphisms, filtrations etc., adapted for computations. This chapter is presented for general sheaves, however, our main example for this is the sheaf of differential forms. In Chapter 4 we prove the Atiyah–Hodge theorem which says that the elements of the de Rham cohomology of an affine variety is given by algebraic differential forms. This paves the road for the definition of algebraic de Rham cohomology in the next chapter. Chapter 5 is fully dedicated to algebraic de Rham cohomology and the fact that it is isomorphic to the classical de Rham cohomology. We need to describe this isomorphism as explicitly as possible because we want to transport the integration of C^∞ forms to the algebraic side, where integration becomes a purely algebraic operation. The objective of this chapter is to collect all necessary material for computing the integration of elements of algebraic de Rham cohomologies over algebraic cycles. In Chapter 6 we capture the cohomology of affine varieties by using logarithmic differential forms. This is needed in order to take residues. This is not possible using only Atiyah–Hodge theorem. This theorem for the complement of smooth hypersurfaces turns out to be the Griffiths theorem which also finds a basis of such de Rham cohomologies. This is explained in Chapter 7. These are used in order to integrate elements of algebraic de Rham cohomology of hypersurfaces over complete intersection algebraic cycles which is done in Chapter 8. Chapter 9 is devoted to the description of Gauss–Manin connection of families of algebraic varieties. In this chapter we work on the general context of arbitrary families of projective varieties, whereas

Movasati (2021) focused on the computation of Gauss–Manin connection for tame polynomials, and in particular families of hypersurfaces. One of the main theorems proved in this chapter is Griffiths transversality. It relates the Gauss–Manin connection to the underlying Hodge filtrations. We do not give concrete applications of the Gauss–Manin connection in Algebraic Geometry, however, its partial data, namely the infinitesimal variation of Hodge structures (IVHS) has successful applications. This includes the famous Noether–Lefschetz theorem which says that a generic surface in the projective space of dimension three has Picard rank one. Chapter 10 is dedicated to this topic. In Chapter 11 we observe that Hodge cycles of smooth hypersurfaces give us Artinian Gorenstein rings, and in this way, many topological problems can be reduced into commutative algebra problems. This will be elaborated more in Chapter 12 in which we explain many well known components of the Hodge locus for hypersurfaces.

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