A dynamical system approach for Lane–Emden type problems

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Preface

The idea of this book and course originated from the experience of a joint project started in 2019, where the first two authors were visiting the university of Rome La Sapienza under the invitation of Prof. Filomena Pacella. At first glance, working with Dynamical Systems seemed challenging, mostly because it was not the area of expertise for any of us. But we developed a huge amount of research, linked to our previous experience, and ended up with a rather good knowledge on the theory. In these notes we present in a simple form the essential tools in Ordinary Differential Equations and Dynamical Systems to solve Partial Differential Equations of fully nonlinear type in the radial regime.

The book is aimed for graduate students interested in Partial Differential Equations and/or Dynamical Systems. We tried to gather a self contained and detailed analysis on the subject, in addition to several references, in such a way to ease the experience of the reader.

Here we exploit the classification of radial positive solutions for a class of fully nonlinear problems. Our approach is entirely based on the analysis of the dynamics induced by an autonomous quadratic system, which is obtained after a suitable transformation. This method allows us to treat both regular and singular solutions in a unified way. It applies to define critical exponents, from which existence and nonexistence of solutions are completely characterized.

It is our goal to enable the reader to identify all trajectories produced by the dynamical system and translate it into positive radial solutions of the corresponding second order partial differential equations problem. We will deal with solutions in a ball, the whole space, exterior domains, and annuli.

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Contents

Introduction

In this book we study existence, uniqueness, nonexistence, and classification of radial positive solutions for some nonlinear problems, subject to a Lane–Emden coupling with Hénon type power weight, and driven by fully nonlinear operators.

Let us recall some history on the development of these problems. Semilinear equations like

$$-\Delta u(x) = f(|x|, u(x)) \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \tag{0.0.1}$$

have long been studied, in special by arising in the context of Astrophysics. Here Δ is the standard Laplacian operator in the Euclidean space $\mathbb{R}^N$,

$$\Delta u \ = \ \textstyle\sum_{i=1}^N e_i \ , \quad \{e_i\}_{i=1}^N = \text{spec}(D^2 u),$$

which is just the sum of the eigenvalues of the Hessian $D^2 u$ whenever u is a C^2 function. In some cases, the admissibility of stationary and spherically symmetric stellar dynamic models is equivalent to the solvability of an equation in the form (0.0.1) when $N = 3$, see (Batt, Faltenbacher, and Horst 1986; Mercuri and Moreira dos Santos 2019; Yi Li 1993) and references therein.

In particular, the Lane–Emden equation in the space

$$-\Delta u(x) = |u(x)|^{p-1} u(x) \quad \text{in} \quad \mathbb{R}^3, \ \ p > 1.$$

refers to the description of certain self-gravitating spherically symmetric stellar systems.

Jonathan Homer Lane (American, 1819–1880) and Robert Emden (Swiss, 1862–1940) were two astrophysicists who lead similar mathematical investigations on stellar structures modeled by polytropic fluids. These were the first stellar models studied in the end of the 19 century (Lane 1870), although are still useful for understanding the basic structure of astrophysical objects. For instance, they apply to stars possessing a spherical symmetry, not varying with time, and without no internal motions. As in (Meier 2012, Section 5.2.4.1), polytropic stellar structures can be derived from Newtonian fluid equations, in which three conservation laws yield to the second order differential equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}u}{\mathrm{d}r}\right) + u^p = 0,$$

such that u satisfies a polynomial relation in the form $\rho = \rho_c\, u^p$, where ρ_c is the central mass density, and p stands for the polytropic index. More general equations like

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^\gamma\,\frac{\mathrm{d}u}{\mathrm{d}r}\right) + r^\sigma u^p = 0,$$

are also called Emden–Fowler in the literature, see the survey (Wong 1975).

The General Relativity theory of Albert Einstein has made impressive and even apocalyptic predictions about the space-time structure of the universe, among them the existence of black holes. Since the important work of the German astronomer Karl Schwarzschild, over the past decades astrophysicists and mathematicians have been devoted to understand their properties, from the ones which possess the same mass as stars up to recent photographed discoveries of supermassive objects lying in the Milky Way's center, by culminating at the Physics Nobel prize awards in 2019, see (Hawking and Ellis 1973; Meier 2012; Overbye and Taylor 2020; Peebles 1972; Rhode 2007; The Event Horizon Telescope Collaboration et al. 2019; Wald 1984).

A related model problem is the so called Hénon equation,

$$-\Delta u(x) = |x|^a\,|u(x)|^{p-1}u(x) \quad \text{in} \quad \mathbb{R}^3. \tag{0.0.2}$$

Michel Hénon (French, 1931-2013) was a mathematician and astronomer who developed an analysis of the stability of spherical steady state stellar systems numerically, in what concerns the concentric shell model, see (Hénon 1973). In the case $a < 0$ this equation is also known as Hardy-Hénon (see for instance (Phan and Souplet 2012)), because its relation with the Hardy inequality by referring to the mathematician Godfrey Harold Hardy (English, 1877-1947).

On the other hand, in this text we will be interested in second order problems with fully nonlinear nature, which means the operator depends on the second derivatives entry in a nonlinear way. In particular, we will look at the following Pucci extremal operators

$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$
$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

for $0 < \lambda < \Lambda$, which is a weighted sum of the eigenvalues $\{e_i\}_{i=1}^N$ of the Hessian D^2u whenever u is a C^2 function. They are named so due to the mathematician Carlo Pucci (Italian, 1925-2003), see (Pucci 1966). They are extremal operators, in the sense that $\mathcal{M}^{+}_{\lambda,\Lambda}(X)$ is the supremum of all linear linear operators in the form $\mathrm{tr}(AX)$ over all matrices with eigenvalues between λ and Λ, while $\mathcal{M}^{-}_{\lambda,\Lambda}$ is the infimum one, see the next chapter for details. In this sense, they define the whole class of fully nonlinear uniformly elliptic operators, see (Caffarelli and Cabré 1995). Hence they play, in the fully nonlinear context, the same role as the Laplace operator in the linear case.

To make matters precise, we study positive radial solutions of the following class of fully nonlinear elliptic equations involving the Pucci's operators,

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + |x|^a u^p = 0, \quad u > 0 \quad \text{in } \Omega, \tag{0.0.3}$$

where $a > -1$, $p > 1$. The set $\Omega \in \mathbb{R}^N$, $N \geqslant 3$, is a radial domain such as $\mathbb{R}^N$, a ball B_R of radius $R > 0$ centered at the origin, the exterior of B_R, or an annulus.

We deal with both regular and singular solutions u of (0.0.3) which are C^2 for $r > 0$. In the singular case Ω will be either $\mathbb{R}^N \setminus \{0\}$ or $B_R \setminus \{0\}$, and we assume the condition

$$\lim_{r\to 0} u(r) = +\infty, \quad r = |x|. \tag{0.0.4}$$

Finally, whenever Ω has a boundary, we prescribe on it the Dirichlet condition

$$u = 0 \text{ on } \partial\Omega, \qquad \text{or} \qquad u = 0 \text{ on } \partial\Omega \setminus \{0\} \text{ under (0.0.4)}.$$

Our solutions are understood in the classical sense out of 0 and they are of class C^1 up to 0, since $a > -1$, as shows our Proposition 3.1.7 ahead.

Let us have in mind the so called dimension-like numbers $\widetilde{N}_\pm$ as

$$\widetilde{N}_+ = \tfrac{\lambda}{\Lambda}(N-1)+1, \qquad \widetilde{N}_- = \tfrac{\Lambda}{\lambda}(N-1)+1\,,$$

whenever $\widetilde{N}_+ > 2$.

A general existence result in bounded domains Ω, not necessarily radial, was obtained in (Quaas and Sirakov 2006) under the condition

$$1 < p \leqslant \tfrac{\widetilde{N}_+}{\widetilde{N}_+ -2} \quad \text{for } \mathcal{M}^+_{\lambda,\Lambda}\,, \qquad 1 < p \leqslant \tfrac{\widetilde{N}_-}{\widetilde{N}_- -2} \quad \text{for } \mathcal{M}^-_{\lambda,\Lambda}$$

These intervals come from the optimal range for existence of supersolutions to $\mathcal{M}^\pm$ when $a=0$, see Theorem 1.2.8 in the next chapter.

When the Pucci's operators reduce to the Laplacian (i.e. for $\lambda = \Lambda$, and $\widetilde{N}_\pm = N$) the previous exponents are equal to $\frac{N}{N-2}$ which is known as the Serrin exponent. They do not provide optimal bounds in terms of solutions of (0.0.3) when $a=0$, which is clear by considering the semilinear case.

Nevertheless, as far as the radial setting is concerned, critical exponents which represent the threshold for the existence of solutions to (0.0.3) can be defined when $N \geqslant 3$. They were introduced for $a=0$ by Felmer and Quaas in the seminal work (Felmer and Quaas 2003) for establishing existence and classification of radial positive solutions in $\mathbb{R}^N$. These are also the watershed for existence and nonexistence of positive solutions in the ball. In the case the dimension is $N=2$ no critical exponent exists for the Laplacian or for the $\mathcal{M}^+_{\lambda,\Lambda}$ operator, while it can still be defined for the $\mathcal{M}^-_{\lambda,\Lambda}$ case, see (Pacella and Stolnicki 2021a). However here we only consider the dimensions $N \geqslant 3$.

Note that every positive solution in the ball when $a=0$ is radial, by (Da Lio and Sirakov 2007), while this is not true in general for $a \neq 0$, even in the semilinear case. In fact, for the Dirichlet problem associated to the standard Hénon equation (0.0.2), in (Willem 2002) it was obtained the existence of a radial solution, in addition to a least energy solution which is not radially symmetric. Moreover, the power weight $|x|^a$ as the parameter a gets large induces symmetry breaking and concentration phenomena, see (Mercuri and Moreira dos Santos 2019; Wang 2006; Yan 2009).

When $\lambda = \Lambda$ the corresponding critical exponents are the same, both in radial and nonradial settings; see (Caffarelli, Gidas, and Spruck 1989) for $a=0$, and (Gladiali, Grossi, and Neves 2013) for $a \neq 0$. The identification of critical exponents in the nonradial case for fully nonlinear operators, in turn, remains open.

For $p>1$ and $a>-1$, we set

$$\alpha = \frac{2+a}{p-1}.$$

In the study of the standard Hénon equation (0.0.2), two classes of radial positive solutions are important: the fast decaying and the slow decaying ones. Namely,

$$\lim_{r\to\infty} r^{N-2}u(r) = c, \quad \text{and} \quad \lim_{r\to\infty} r^{\alpha}u(r) = c,$$

respectively, for some $c > 0$ whenever $N > 2$. It is known that fast decaying solutions exist at the critical exponent

$$p^a_\Delta = \frac{N+2+2a}{N-2},$$

while slow decaying solutions emerge for $p > p^a_\Delta$.

In the aforementioned work (Felmer and Quaas 2003) it was shown for the operators $\mathcal{M}^\pm$, when $a = 0$, and $\lambda < \Lambda$, the existence of critical exponents $p^*_\pm$ as far as $\tilde{N}_+ > 2$. They play the role of p^a_Δ for Laplacian in the sense of being the threshold for existence and nonexistence of regular solutions in $\mathbb{R}^N$, see Theorem 1.2.9 in the next chapter. A new class of solutions was also detected, namely pseudo-slow decaying solutions, which satisfy

$$c_1 = \liminf_{r\to\infty} r^{\alpha}u(r) < \limsup_{r\to\infty} r^{\alpha}u(r) = c_2 \ ,$$

for some $0 < c_1 < c_2$, see Definition 1.2.3.

In the case of the operator $\mathcal{M}^+$ the authors also made precise the range of the exponent p for which pseudo-slow decaying solutions exist. The proof of this result in (Felmer and Quaas 2003) is involved. It is a combination of the Emden–Fowler phase plane analysis and the Coffman Kolodner technique. The latter consists in differentiating the solution with respect to the exponent p, and then studying a related nonhomogeneous differential equation, from which they derive the behavior of the solutions for p on both right and left hand sides of $p^*_\pm$, as well as the uniqueness of the exponent p for which a fast decaying solution exists.

The existence of a critical exponent unveils an important feature of the Pucci's operators. It reflects some intrinsic properties of these operators and induces concentration phenomena besides of energy invariance, see (Birindelli et al. 2018), as it happens in the classical semilinear case.

In this book we show how to derive critical exponents, for both regular and singular solutions, of the respective weighted problem (0.0.3), in light of our recent paper (Maia, Nornberg, and Pacella 2020). In what concerns regular solutions our results are similar to those in (Felmer and Quaas 2003), a little bit improved, but with the difference that we exploit much more the strength provided by the dynamical system itself, which makes the proofs simpler. This approach has further been used in (Pacella and Stolnicki 2021a) to refine the bounds given in (Felmer and Quaas 2003) for the critical exponents even more.

Another advantage of our approach is that it treats in a unified way several kind of solutions to (0.0.3). We refer the reader to Figures 4.1 to 4.7 where, for a given value of the exponent p, all the orbits of the system corresponding to different types of solutions of (0.0.3) are displayed simultaneously. Our proofs do not involve energy functions, except for showing a center configuration which appears at the critical exponents, in addition to a particular existence result in annuli for weighted equations.

We highlight that the strategy of introducing an associated quadratic system to treat radial Lane–Emden problems, with or without weight, has long been managed, see for instance (Chicone and Tian 1982; Wong 1975) and references therein. We will follow more closely the ideas applied in (Bidaut-Véron and Giacomini 2010). Finally, we point out that the quadratic system approach can be extended to treat systems with Lane–Emden configuration, in the spirit of (Bidaut-Véron and Giacomini 2010). In the fully nonlinear context, it is the subject of our recent work (Maia, Nornberg, and Pacella 2021).

The text is organized as follows. In Chapter 1 we introduce notations and give an overview of the prerequisites on Dynamical Systems and second order partial differential equations that will be used throughout the text. In particular, in Section 1.2.1 we write down equation (0.0.3) in the radial form. In Chapter 2 we introduce the quadratic system associated to (0.0.3), and study its intrinsic flow properties. In Chapter 3 we classify the different solutions of (0.0.3) in terms of orbits of the corresponding dynamical systems. In the last chapter, Sections 4.1 and 4.2 are devoted to the proofs of the main results for the Pucci $\mathcal{M}^+$ and $\mathcal{M}^-$ operators, respectively.

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