Visualizing Thurston's geometries

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Preface

In 2018, NVidia introduced the RTX series of GPUs enabling the implementation of real-time ray-tracing algorithms in Euclidean spaces, which allows visualization applications with a high degree of photorealism. In the same year, Luiz Velho and Vinícius da Silva started the Ray-VR project, at IMPA's Visgraf Laboratory, to integrate Virtual Reality (VR) and ray tracing. In 2019, Tiago Novello joined the project, which started using the developed framework to visualize non-Euclidean spaces in an immersive and interactive way.

Examples of non-Euclidean spaces date back to Thurston's geometrization conjecture, which states that any three-dimensional compact manifold one decomposes into geometrically modeled pieces by just eight geometries. Ray-VR gave rise to a system for immersive and interactive visualization (in VR) of spaces modeled by the classic Thurston geometries (Euclidean, spherical, and hyperbolic), later, the results were extended to manifolds modeled by the ”twisted” geometries (Nil, Sol, and SL2): the least trivial Thurston's geometries.

This book presents a compilation of Ray-VR results in the intrinsic visualization of Thurston's geometries. This is an active research topic in mathematical visualization that combines the areas of geometry and topology, with concepts of computer graphics. The content of this book serves both experts and students. Although this is a short book, it is self-contained since it considers all the ideas, motivations, references, and intuitive explanations of the required fundamental concepts.

It is important to highlight that several conditions made this a special moment for such a topic. On one hand, the development of mathematical research and graphics algorithms has provided the theoretical framework. On the other hand, the evolution of media technologies allows us to be immersed in three-dimensional spaces using VR.

The target reader of this book would be interested in geometry, topology, mathematical education, and also interested in new visualization techniques to explore abstract spaces. For the public interested in media, this work offers the possibility to explore new mathematical scenarios. These visualizations have the potential to be applied in entertainment, arts, education, cinema, and games.

We thank the Organizing Committee of the 33rd Brazilian Colloquium of Mathematics for the opportunity to present our results in mathematical visualization.

1 Background on Manifolds and Orbifolds

1.1 History

This chapter sets the stage with an historical account of the quest to investigate 2D and 3D spaces, as well as the context related to the Poincaré Conjecture inspiring the classification / geometrization of compact two and three dimensional manifolds.

1.1.1 Henri Poincaré

In 1895, Henri Poincaré published his *Analysis situs* (Poincaré 1895), in which he presented the foundations of topology by proposing to study spaces under continuous deformations; position is not important. The main tools for topology are introduced in this paper: manifolds, homeomorphisms, homology, and the fundamental group. He also discussed about how three-dimensional geometry was real and interesting. However, there was a confusion in his paper: Poincaré treated homology and homotopy as equivalent concepts.

In 1904, Poincaré wrote the fifth supplement 1904 to Analysis situs, where he approached three-dimensional manifolds. This paper clarified that homology was not equivalent to homotopy in dimension three. He presented the *Poincaré dodec-*

ahedron as an example of a 3-manifold with trivial homology but with nontrivial homotopy. In Section 1.4.1, we present an inside view of such space. Poincaré proposed the conjecture: Is the 3-sphere the unique compact connected 3-manifold with trivial homotopy?

Poincaré stimulated a lot of mathematical works asking whether some manifold exists. Works on this question were awarded three Fields medals. In 1960, Stephen Smale proved 2007 the conjecture for n-manifolds with $n > 4$. In 1980, Michael Freedman proved 1982 Poincaré conjecture for 4-manifolds. The problem in dimension three was open until 2003 when Grigori Perelman proved (Perelman 2002, 2003a,b) Thurston's geometrization conjecture, and consequently the Poincaré conjecture as a corollary.

Poincaré also worked on an important problem in dimension two, the *uniformization theorem*. This states that every simply connected *Riemann surface* (one-dimensional complex manifolds) is *conformally equivalent* to the unit disc, the complex plane, or the Riemann sphere. This was conjectured by Poincaré in 1882 and Klein in 1883, and proved by Poincaré and Koebe in 1907. The history details can be found in the recent book by Ghys (2017). A big step in the history of the geometry was the generalization of this result for dimension three, *Thurston's geometrization theorem*.

1.1.2 William P. Thurston

Thurston's works in 3-manifolds have a geometric taste with roots in topology. He tried to generalize the uniformization theorem of compact surfaces to dimension three. Five more geometries arise; the hyperbolic still playing the central role.

In 1982, Thurston stated the *geometrization conjecture* (Thurston 1982) with solid justifications. It is a three-dimensional version of the uniformization theorem, where hyperbolic geometry is the most abundant because it models all surfaces with genus greater than one. In dimension three, Thurston (ibid.) proved that the conjecture holds for a huge class of 3-manifolds, the *Haken manifolds*, implying that hyperbolic plays, again, the central role. The result is known as the *hyperbolization theorem*. Thurston received in 1982 a Fields medal for his contributions to 3-manifolds. The *Elliptization conjecture*, the part which deals with spherical manifolds, was open at that time.

1.1.3 Grigori Perelman

In 2000, the Clay Institute selected seven problems in mathematics to guide mathematicians in their research, the *seven Millennium Prize Problems* (Jaffe 2006). Poincaré conjecture was one of them. They did not know that the Poincaré conjecture was about to be solved by Grigori Perelman as a corollary of the proof of the geometrization conjecture.

In 2003, Perelman published three papers (Perelman 2002, 2003a,b), in arXiv solving the Geometrization conjecture. He used tools from geometry and analysis. Specifically, he used the *Ricci flow*, a technique introduced by Richard Hamilton to prove the Poincaré conjecture. Hamilton proved the conjecture for a special case when the 3-manifold has positive *Ricci curvature*. The idea is to use Ricci flow to simplify the geometry along time. However, this procedure may create *singularities* since this flow expands regions with negative Ricci curvature and contracts regions of positive Ricci curvature. Hamilton suggested the use of *surgery* before the manifold collapse. The procedure gives rise to a simpler manifold, and we can evolve the flow again. Perelman, proved that this algorithm stops and each connected component of the resulting manifold admits one of the Thurston geometries. In other words, Perelman proved the geometrization conjecture, and consequently the Poincaré conjecture.

1.2 2-Manifolds

We present some results involving topology and geometry of surfaces. We assume all surfaces been compact, connected, and oriented. Starting with the *classification theorem* in terms of the *connected sum*, one can represent a surface through a polygon with an appropriate edge gluing. This polygon can be embedded in one of the three two-dimensional geometry models (Euclidean, spherical, and hyperbolic). The resulting surface has the geometry modeled by one of these geometries.

1.2.1 Classification of compact surfaces

The classical way to state the classification theorem is by the *connected sum*. Removing disks D_1 and D_2 from surfaces S_1 and S_2, one obtains their connect sum $S_1\#S_2$ by identifying the boundaries ∂D_1 and ∂D_2 through a homeomorphism. The theorem says that any compact surface is homeomorphic to a sphere or a connected sum of tori. The theorem proof uses a computational representation of a compact surface S through an appropriate pairwise gluing of edges in a polygon:

- Take a triangulation T of S; it is a well-known result;
- Cutting along edges in T we obtain a list of triangles embedded in the plane without intersection; the edge pairing must be remembered;
- We label each triangle edge with a letter according to its gluing orientation;
- Gluing the triangles through their pairwise edge identifications without leaving the plane produces a polygon P. The boundary ∂P is an oriented sequence of letters;
- Let a and b be a couple of edges in ∂P. If the identification of a and b reverses the orientation of ∂P we denote b by a^{-1}, and simply a otherwise;
- A technical result states that by cutting and gluing P leads us to an equivalent polygon Q with its boundary having one of following configurations:
  - aa^{-1}, which is a sphere;
  - $\sum aba^{-1}b^{-1}$, a connected sum of tori $aba^{-1}b^{-1}$.

To model the geometry of those surfaces, we embed, in a special way, the polygon in one of the two-dimensional model geometries.

1.2.2 Geometrization of compact surfaces

We remind the well-known *geometrization* theorem of compact surfaces which states that any topological surface can be modeled using only three geometries.

Theorem 1 (Geometrization of surfaces)**.** *Any compact surface admits a geometric structure modeled by the Euclidean, the hyperbolic, or the spherical space.*

The Euclidean space $\mathbb{E}^2$ models the geometry of the 2-torus through the quotient of $\mathbb{E}^2$ by the group of translations. The sphere is modeled by the spherical geometry.

For a hyperbolic surface, consider the *bitorus*, which topologically is the connect sum of two tori. The bitorus is presented as a regular polygon P with 8 sides $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ as discussed above. All vertices in P are identified into a unique vertex v. Then, the 8 corners of P are glued together producing a topological disk. Considering P with the Euclidean geometry, the angular sum around v equals to 6π. To avoid such a problem, let P be a regular polygon centered in the hyperbolic plane, with an appropriate scale, its angles sum $\pi/4$. The edge pairing of P induces a group action Γ in the hyperbolic plane $\mathbb{H}^2$ such that $\mathbb{H}^2/\Gamma$ is the bitorus. In terms of *tessellation*, Γ tessellates $\mathbb{H}^2$ with regular 8-gons. Analogously, all surfaces represented as polygons with more than four sides are hyperbolic. Implying that hyperbolic is the most abundant geometry.

The above discussion handled all orientable surfaces. The well-known Gauss–Bonnet theorem implies that these geometric structures must be unique.

1.3 3-Manifolds

It took time to formulate the modern idea of a manifold in a higher dimension. For example, a version of Theorem 1 for 3-manifold seemed not possible until 1982, when Thurston proposed the geometrization conjecture 1982. There are exactly eight geometries in dimension 3, which are presented in more detail in Section 1.4. Scott (1983) is a great text on this subject.

1.3.1 Classification of compact 3-manifolds

As for surfaces, there is a combinatorial procedure to build three-dimensional manifolds from identifications of polyhedral faces. To do so, endow a finite number of polyhedra with an appropriate pairwise identification of its faces. Each couple of faces has the same number of edges and it is mapped homeomorphically to each other. Such gluing gives a *polyhedral complex* K, which is a 3-manifold iff its Euler characteristic is equal to zero (Theorem 4.3 in (Fomenko and Matveev 2013)).

We now take the opposite approach. Let M be a compact 3-manifold, we represent M as a polytope P endowed with a pairwise identification of its faces. The following algorithm mimics the surface case presented in Section 1.2.1.

- Let T be a triangulation of M; endorsed by the triangulation theorem;
- Detaching every face identification in T gives rise to a collection of tetrahedra which can be embedded in $\mathbb{E}^3$. Remember the pairwise face gluing;
- Gluing in a topological way each possible coupled tetrahedra without leaving $\mathbb{E}^3$ produces a polytope P. The faces in ∂P are pairwise identified.

The combinatorial problem of reducing P to a standard form, as in the surface case, remains open (see page 145 in Lee (2010)). Although there is not (yet) a classification of compact 3-manifold in the sense presented for compact surfaces,

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