An introduction to Characteristic Classes

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33º Colóquio Brasileiro de Matemática

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Primeira impressão, julho de 2021

Publicado no Brasil / Published in Brazil.

ISBN 978-65-89124-15-3
MSC (2020) Primary: 14C17, Secondary: 55R40, 57R20, 57R25, 58K45

Coordenação Geral Carolina Araujo

Produção Books in Bytes **Capa** Izabella Freitas & Jack Salvador

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ

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editora@impa.br

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Preface

Defining the birth of characteristic classes is not clear.

Who of Pythagoras, Plato, Maurolico, Descartes, Euler, Poincaré, Hopf... can be considered as the creator of the characteristic classes?

This is the reason why I invite you during the course for a cruise in which we will meet these people and others... who will share their contribution with us.

Our cruise starts on the island of Samos, Greece in 570 BC where we meet Pythagoras playing with representations of the tetrahedron, the hexahedron (cube) and the octahedron. Thales comes from Miletus not far from the island to play with us. We leave the island for Athens where we meet Theaetetus, less known than Plato, although he is the finder of the 5 platonic polyhedra.

After playing with the polyhedra, through the Mediterranean Sea we leave Greece for Sicily where we have a walk on the Syracuse beach with Archimedes in 230 BC. Still in Sicily, in Messina, much later, in December 26, 1537, we meet an Italian priest, Francesco Maurolico who, apparently does not care of the war between Charles V and the Pope against the Turks and prefers to spend his time describing the planar representations of the platonic polyhedra. Maurolico tells us that he observed that the 5 platonic polyhedra satisfy the formula:

$$\#\text{vertices } - \#\text{edges } + \#\text{faces} = +2.$$

The boat takes us to Stockholm, in January 1650, where Descartes is invited by the Queen Christina of Sweden. Descartes is very ill. He entrusts us with his manuscripts, containing among other things a "nice theorem". When Descartes dies, few days later, we take the boat which transports his manuscripts to Paris in a safe. Arriving in Paris, the boat sinks (do you know how to swim?). Fortunately, after 3 days in the river Seine, the safe is recovered, allowing later Leibniz to copy Descartes' manuscripts and to take these copies to Hanover in Germany.

Still in Germany, in 1750 we go to the Jean-Sebastien Bach's funerals. Then in Berlin, on November 14, we meet Euler who just sent a letter to his friend Goldbach saying that he discovered the formula that now bears his name, this for all convex polyhedra in $\mathbb{R}^3$. The formula appears now in fact as a corollary of the theorem that Descartes showed us.

We return in Paris, in 1885, for the Victor Hugo's funerals. Poincaré is there and he says us that he has been able to generalize the Euler characteristic for all dimensions. We stay in Paris to follow the construction of the Eiffel Tower and of the first metro line. We meet again Poincaré in 1899 who tells us that the characteristic, now called Euler–Poincaré, is the obstruction to the construction of a vector field tangent to a compact smooth surface.

Back in Berlin, in 1927, we go to the cinema to see the new silent film "Metropolis" by Fritz Lang. We are sitting next to Hopf who invites us to know how he generalizes the Poincaré result for all dimensions, obtaining the now called Poincaré–Hopf theorem.

Hopf advises his student Stiefel to study the obstruction to construct an r-frame tangent to a smooth manifold. For us, we continue our cruise, this time on the liner "Normandie", on May 29, 1935 for its inaugural crossing to USA. In a festive atmosphere, he wins the "blue ribbon". We reach Weston, where Whitney shows us around his marvelous house. He takes us to climb the peaks of Massachusetts (don't you feel dizzy?). Whitney tells us that he has a similar construction to the Stiefel's one. The Stiefel–Whitney classes are born.

We stay in the United States during WWII and Chern invites us to Princeton in 1946 to read us his poems and tell us how he constructs, in the complex setting, the "Chern classes" in so many ways, not just using the obstruction theory but also, among others, by decomposition of Grassmannian manifolds into Schubert cycles and by differential forms. Chern gets us so excited that we forget our cruise. He suggests us to follow evolution of the theories: Chern–Gauß–Bonnet, Chern–Weil, Chern–Simons.

Back in France, in 1965, we take the train from Paris to Lille in which we meet a woman making strange drawings on sheets of paper. Marie-Hélène Schwartz explains to us that she understood why, in general, the Poincaré–Hopf Theorem does not work for singular varieties and, radiant, she explains to us that we must consider radial fields, making the picture of a pinched torus and folding another sheet of paper. Moreover, she tells us to be able to define Chern classes for singular varieties, using what she calls "Whitney stratifications".

Four years later, in IHES in Paris, Deligne and Grothendieck conjecture existence and uniqueness of Chern classes for singular varieties, satisfying a system of axioms. The conjecture is proved by Robert MacPherson in 1973. In Paris, Marie-Hélène Schwartz enters in a small clothing store on the Boulevard Saint Michel to buy a shirt for her husband, Laurent. By chance, MacPherson also walks into the same cramped store. No place for us, but we can hear the discussion about characteristic classes ending by a "they must be the same". They are the same and that has been proved by Marie-Hélène Schwartz and myself. We use one ingredient, defined by MacPherson, the "local Euler obstruction".

Now, we go to Brasil ! MacPherson gave a lecture during the 9th Colóquio Brasileiro de Matemática, that was in 1973 in Poços de Caldas. We are now at the... 33th Colóquio Brasileiro de Matemática ! The cruise is not finished, there are many researchers, women and men, working on the characteristic classes and on the developments of the local Euler obstruction in Brasil and other countries. The story will continue, with you ?

Readers interested in deepening their knowledge in the subject of characteristic classes can consult, for the smooth case, the books by Dieudonné (1989) and Steenrod (1951), the book by Milnor and Stasheff (1974) "characteristic classes" and, for the case of singular varieties, the forthcoming article by the author to appear in the Handbook of Geometry and Topology of Singularities volume III. Springer.

Eu dedico o curso a Roberto Callejas-Bedregal, falecido do covid em Abril. Jamais esqueceremos sua alegria comunicativa de trabalhar em matemática.

Agradeço ao 33º comitê CBM por me dar a honra de ministrar este curso. Agradeço a todas as pessoas que me ajudaram a escrever o livro do curso e a fazer os vídeos: matemáticos, responsáveis, secretárias, administradores, editores e técnicos.

Muito obrigado a Carolina, Thủy, Paulo, Suely, Leticia, Anderson,…Obrigado a todos por sua ajuda.

Jean-Paul Brasselet
Julho de 2021

1 Introduction

1.1 Manifolds and pseudomanifolds

In the following, we recall basic, and useful, notions about manifolds. In this section and the following ones, unless explicit mention, all considered (pseudo) manifolds are connected.

One of the fundamental notion that we will use is the one of triangulation of the considered spaces, that is done through the notion of simplicial complex.

A simplex is the convex hull of $(k+1)$ linearly independent points in the euclidean space $\mathbb{R}^m$. A ℓ-dimensional face of the simplex is the convex hull of $(\ell+1)$ of these points, $\ell \leqslant k$.

Definition 1.1.1. A (finite) simplicial complex K is a collection of simplexes in some euclidean space $\mathbb{R}^m$ such that

- if $s \in K$ then every face of s belongs to K,
- if $s, t \in K$, then $s \cap t$ is either empty or is a common face of s and t.

Definition 1.1.2. Let us denote by K a (finite) simplicial complex in $\mathbb{R}^m$. The union of simplexes in K is a compact subspace of $\mathbb{R}^m$ denoted by $|K|$ and called *geometric realisation* of K, or *polyhedron* associated to K.

Definition 1.1.3. A topological space X is *triangulable* (or a polyhedron) if there exists a simplicial complex K and a homeomorphism $h : |K| \to X$. Such a pair (K, h), or simply the simplicial complex K, is called a *triangulation* of X.

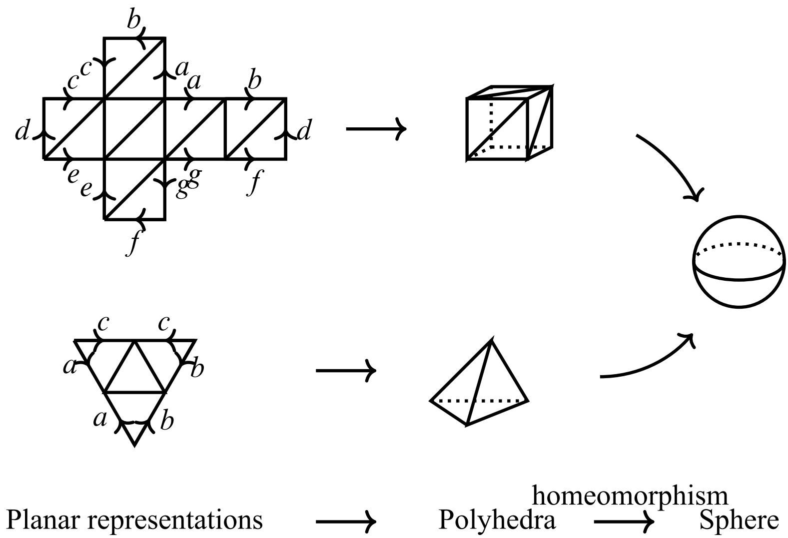

Figure 1.1: Triangulations of the sphere.
In the planar representations, the segments with same name are identified respecting the orientation.

Remark 1.1.4. Not all topological spaces are triangulable (seeVerona (1984)).

Let K be a simplicial complex, and x a point in the polyhedron $|K|$. The simplicial neighbourhood of x in K, denoted by $N_K(x)$ is the set of (closed) simplexes that contain x together with their faces. The link of x, denoted by $\mathrm{Lk}_K(x)$ is the subset of simplexes in $N_K(x)$ that do not contain x. The i-skeleton of K, denoted by $K^{(i)}$, is the set of (K)-simplices whose dimension is less or equal to i.

Definition 1.1.5. (Combinatorial manifold) A polyhedron $|K|$ is called a *combinatorial n-manifold* if for each $x \in |K|$, the link $|\mathrm{Lk}_K(x)|$ is homeomorphic to the sphere $\mathbb{S}^{n-1}$.

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ISBN 978-65-89124-15-3

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